ON k-PERIODIC SYSTEMS OF LINEAR EQUATIONS

BY

I. M. SHEFFER

1. **Introduction.** We shall be interested, in this work, in certain cases of the system of equations

$$\mathcal{A}: \quad a_{n0}x_n + a_{n1}x_{n+1} + \cdots = c_n \qquad (n = 0, 1, 2, \cdots) \tag{1.1}$$

in the infinitely many unknowns $x_0, x_1, \cdots$, cases that we shall refer to as *k-periodic systems, with or without perturbation terms*. System (1.1) is characterized by the coefficient array $\{a_{nj}\}$, or equally well by the sequence of functions $\{A_n(t)\}$, $n=0, 1, \cdots$, defined by

$$A_n(t) = \sum_{j=0}^{\infty} a_{nj}t^j \qquad (n = 0, 1, \cdots). \tag{1.2}$$

In the cases that concern us the series (1.2) will have nonzero radii of convergence.

DEFINITION. A system $\mathcal{A}$ of form (1.1) is a *k-periodic system* if the coefficients repeat themselves in blocks of k rows; in other words, if

$$A_{i+jk}(t) = A_i(t) \qquad (i = 0, 1, \cdots, k-1; j = 0, 1, \cdots). \tag{1.3}$$

A k-periodic system has the form

$$\mathcal{A}: \quad \sum_{r=0}^{\infty} a_{i,r}x_{i+jk+r} = c_{i+jk} \qquad (i = 0, 1, \cdots, k-1; j = 0, 1, \cdots). \tag{1.4}$$

Such a system is completely determined by knowledge of the k functions, $A_0(t), \cdots, A_{k-1}(t)$. Observe that a k-periodic system need not have k as a *primitive period.*

It is time to say something about the kind of solution that we seek. To this end we state the following definition.

DEFINITION. By the *type* of a sequence $\{x_n\}$ is meant the number $((x_n))$ defined by

$$((x_n)) \equiv \limsup_{n\to\infty} |x_n|^{1/n}. \tag{1.5}$$

It is clear that if the functions $A_j(t)$, $j=0, 1, \cdots, k-1$, are analytic in $|t| \leqq q$, then the series on the left of (1.4) all converge if $((x_n)) \leqq q$; in fact, convergence takes place when $((x_n)) < p$, where p is the smallest radius of convergence of functions $\{A_n(t)\}$. We shall look for solutions of type not

Presented to the Society, August 23, 1946; received by the editors February 20, 1947.

exceeding q. For sequences $\{x_n\}$ with $((x_n)) \leqq q$, it is readily found that $\{c_n\}$ given by (1.4) satisfies the condition $((c_n)) \leqq q$; so this condition will be imposed on $\{c_n\}$.

The k-periodic system *with perturbations* is formed from (1.4) by adding in terms with coefficients $\{a^*_{nj}\}$:

$$(1.6) \qquad \mathcal{A} + \mathcal{A}^*: \quad \sum_{r=0}^{\infty} (a_{i,r} + \overset{*}{a}_{i+jk,r}) x_{i+jk+r} = c_{i+kj}$$

$$(i = 0, 1, \cdots, k-1; j = 0, 1, \cdots),$$

where the numbers $\{a^*_{nj}\}$ are small relative to $\{a_{nj}\}$. The precise conditions to be placed on $\{a_{nj}\}$ and $\{a^*_{nj}\}$ will be specified in §§2 and 5 respectively.

System (1.6), *for the case* $k=1$, arose in extending and precising a work of Poincaré(1) on certain difference equations, and was given a complete and elegant treatment by Perron(2)(3). The treatment of the general case in the present paper uses the methods of Perron where possible; but these methods must be augmented by new ones in order to carry out successfully the present program.

In §2 it is shown how a system of type (1.1) can be transformed by means of a *G-transformation* into a new system. When applied to a k-periodic system, it gives rise to a determinant function $\Delta_A(t)$ whose zeros (which occur in nests of k zeros each) are shown eventually to determine the number of solutions of the original system. More immediately, however, the G-transformation carries the system into a one-periodic system, to which the Perron theory applies. But this new system is not in general equivalent to the original: it usually has more solutions than has the original, and it is no easy matter to sort out the true solutions from the overabundance of solutions given by the new system.

To achieve this aim of obtaining the true solutions, we turn in §3 to the problem of *factoring* a k-periodic system $\mathcal{A}$. Such factorization is always possible, and it is shown that each factor has a determinant function related to $\Delta_A(t)$ but much simpler in that its zeros consist of a single nest taken from the nests possessed by $\Delta_A(t)$. By this factorization process the true number of linearly independent solutions of (1.4) is determined as the number of nests (of zeros) of $\Delta_A(t)$.

In §4 certain algebraic properties of factoring are examined, and this leads to a new, simpler representation of the k-periodic system $\mathcal{A}$ as a product of factors. The case of a system with perturbations is the subject matter of the

(1) H. Poincaré, *Sur les equations linéaires aux différentielles ordinaires et aux différences finies*, Amer. J. Math. vol. 7 (1885) pp. 203–258.

(2) O. Perron, *Über Summengleichungen und Poincarésches Differenzengleichungen*, Math. Ann. vol. 84 (1921) pp. 1–15.

(3) Milne-Thomson, *The calculus of finite differences*, London, Macmillan, 1933. Chapter 17 is given over to an English translation of the paper of Perron.

next two sections. In §5 it is shown that a perturbation system can also be factored in the case that no two nests of zeros of $\Delta_A(t)$ have the same absolute value. From this factorization it is then possible to solve completely the corresponding system of equations (1.6), by treating the simple factors separately. Finally, in §6, without benefit of the factorization of §5, a complete solution of system (1.6) is found ***in every case***; but lacking this factorization, the work is correspondingly more complicated.

2. **The G-transformation.** Consider the general system of form (1.1) (and therefore not necessarily periodic):

$$\mathcal{A}: \quad \sum_{j=0}^{\infty} a_{nj}x_{n+j} = c_n \qquad (n = 0, 1, \cdots), \tag{2.1}$$

and let $\{g_{ij}\}$ be an array of constants. Multiply the equations of (2.1) respectively by $g_{00}, g_{01}, \cdots, g_{0n}, \cdots$ and add. Then multiply equations (2.1) (save the 0th) by $g_{10}, g_{11}, \cdots$ and add; then (save the 0th and 1st) by g_{20}, $g_{21}, \cdots$, and so on. There results the system

$$\mathcal{A}_G: \quad \sum_{j=0}^{\infty} h_{nj}x_{n+j} = c_n', \qquad n = 0, 1, \cdots, \tag{2.2}$$

where formally

$$c_n' = \sum_{j=0}^{\infty} g_{nj}c_{n+j} \tag{2.3}$$

and

$$H_n(t) \equiv \sum_{j=0}^{\infty} h_{nj}t^j = \sum_{j=0}^{\infty} g_{nj}t^jA_{n+j}(t). \tag{2.4}$$

Here $\{A_n(t)\}$ is defined by

$$A_n(t) = \sum_{j=0}^{\infty} a_{nj}t^j. \tag{2.5}$$

System $\mathcal{A}_G$, given by (2.2), we term the *G-transform*(4) of system $\mathcal{A}$.

So far the G-transform is purely formal; the series defining $\{c_n'\}$ need not even converge. The following theorem gives a sufficient condition for the validity of the transformation.

THEOREM 2.1. *Let the functions $\{A_n(t)\}$, $\{G_n(t)\}$ be analytic in $|t| \leqq q$, where $\{A_n(t)\}$ is given by* (2.5) *and $\{G_n(t)\}$ by*

$$G_n(t) = \sum_{j=0}^{\infty} g_{nj}t^j \qquad (n = 0, 1, \cdots). \tag{2.6}$$

---

(4) The G-transform can also be applied to the system $\sum_{j=0}^{\infty} a_{nj}x_j = c_n$, $n = 0, 1, \cdots$, but this does not concern us here.

To every $\epsilon>0$ let there correspond a positive number $\delta=\delta(\epsilon)$ such that

$$(2.7)\qquad ((A_n^*(q+\delta)))<1+\epsilon,\qquad ((G_n^*(q+\delta)))<1+\epsilon,$$

where

$$(2.8)\qquad A_n^*(t)=\sum_0^\infty |a_{nj}|\,t^j,\qquad G_n^*(t)=\sum_0^\infty |g_{nj}|\,t^j.$$

Then for all $\{x_n\}$ with $((x_n))\leqq q$, the series on the left of (2.1) *converge to a sequence $\{c_n\}$ for which $((c_n))\leqq q$; sequence $\{c_n'\}$ of* (2.3) *also exists and is of type not exceeding q; and $\{x_n\}$ satisfies* (2.2). *Moreover, the functions $\{H_n(t)\}$ are analytic in $|t|\leqq q$, and have the further property that*

$$((H_n^*(q+\lambda)))<1+\epsilon,$$

where $H_n^*(t)=\sum_0^\infty |h_{nj}|\,t^j$ and $\lambda=\lambda(\epsilon)$.

REMARK. Theorem 2.1 shows that system (2.2) is, under appropriate conditions, a consequence of (2.1). In general, however, we shall see later that system (2.1) is *not* implied by (2.2): the two systems are not equivalent.

We proceed to the proof of Theorem 2.1. Let $((x_n))\leqq q$. To $\epsilon_1>0$ corresponds a K_1 such that $|x_n|\leqq K_1(q+\epsilon_1)^n$, and to $\epsilon_2>0$ correspond δ_2 and M_2 such that $A_n^*(q+\delta_2)<M_2(1+2\epsilon_2)^n$. Using the Cauchy inequality for power series coefficients, we find that $|a_{nj}|\leqq M_2(1+2\epsilon_2)^n\cdot(q+\delta_2)^{-j}$, so (for ϵ_1 chosen less than δ_2) $\sum_{j=0}^\infty |a_{nj}x_{n+j}|$ converges to a sum not exceeding $C[(q+\epsilon_1)\cdot(1+2\epsilon_2)]^n$ (C=constant). This shows that sequence $\{c_n\}$ exists and that $((c_n))\leqq q$.

One likewise establishes that $|g_{nj}|\leqq M_3(1+2\epsilon_3)^n(q+\delta_3)^{-j}$, where $\epsilon_3>0$ is arbitrary and M_3, δ_3 are positive numbers depending on ϵ_3, and that $\{c_n'\}$ exists, with $((c_n'))\leqq q$. It is then a straightforward argument to establish the remaining assertions of the theorem.

Now consider the *k-periodic system*

$$(2.9)\qquad \mathcal{A}:\ \sum_{r=0}^\infty a_{ir}x_{i+jk+r}=c_{i+jk}\qquad (i=0,1,\cdots,k-1;\ j=0,1,2,\cdots),$$

where the functions $A_0(t),\cdots,A_{k-1}(t)$ are analytic in $|t|\leqq q$ ($q>0$), and where

$$(2.10)\qquad A_j(0)\neq 0\qquad (j=0,1,\cdots,k-1).$$

We introduce the *determinant function*

$$(2.11)\qquad \Delta_A(t)=\begin{vmatrix} A_0(t) & A_1(t) & A_2(t) & \cdots & A_{k-1}(t)\\ A_0(\omega t) & \omega A_1(\omega t) & \omega^2A_2(\omega t) & \cdots & \omega^{k-1}A_{k-1}(\omega t)\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ A_0(\omega^{k-1}t) & \omega^{k-1}A_1(\omega^{k-1}t) & \omega^{2(k-1)}A_2(\omega^{k-1}t) & \cdots & \omega^{(k-1)(k-1)}A_{k-1}(\omega^{k-1}t)\end{vmatrix},$$

where ω is a kth root of unity:

$$\omega = e^{2\pi i/k} \qquad (i = (-1)^{1/2}). \tag{2.12}$$

COROLLARY 2.1. *The quantity* $\Delta_A(0)$ *is not zero.*

For,

$$\Delta_A(0) = A_0(0) \cdots A_{k-1}(0) \cdot V,$$

where V is a Vandermond determinant in the k distinct quantities $1, \omega, \cdots, \omega^{k-1}$. We also have the following theorem.

THEOREM 2.2. *For* $f=0, 1, \cdots, k-1$,

$$\Delta_A(\omega^f t) = \Delta_A(t), \tag{2.13}$$

so the power series expansion of $\Delta_A(t)$ *contains only powers of* t^k.

It suffices to establish (2.13) for $f=1$. Multiply the jth column of $\Delta_A(t)$ by ω^j, $j=0, 1, \cdots, k-1$, obtaining a determinant of value $\omega^m \Delta_A(t)$ with $m=k(k-1)/2$. Replace t by ωt in every element of the new determinant. It is then seen that $\omega^m\Delta_A(\omega t)$ is the value of a determinant obtained from $\Delta_A(t)$ by shifting each row up one (the top row going to the bottom). Hence $\omega^m\Delta_A(\omega t) = (-1)^{k-1}\Delta_A(t)$, and since $\omega^m = (-1)^{k-1}$ we conclude that $\Delta_A(\omega t) = \Delta_A(t)$.

Let $P(t)$ be the *polynomial of zeros* of $\Delta_A(t)$ in $|t| \leqq q$. That is, $P(t) \equiv 1$ if $\Delta_A(t)$ has no zeros in $|t| \leqq q$; and if Δ_A has the zeros $\alpha_1, \cdots, \alpha_m$ there (multiple zeros counted multiply), then

$$P(t) \equiv \prod_{j=1}^{m} (t - \alpha_j).$$

We may then write

$$\Delta_A(t) = P(t)\Delta^*(t)$$

where $\Delta^*(t)$ is analytic and without zeros in $|t| \leqq q$. From (2.13) we deduce the following corollary.

COROLLARY 2.2. *For* $f=0, 1, \cdots, k-1$,

$$P(\omega^f t) = P(t). \tag{2.14}$$

The zeros of $P(t)$ (which is to say of $\Delta_A(t)$) can therefore be grouped into nests of k each, such that if α is a zero, then the other zeros of the same nest are $\omega^f\alpha$, $f=1, \cdots, k-1$. And if α is of multiplicity p, then α occurs in precisely p nests, and every pair of these nests contain the same zeros. The total number m of zeros of $\Delta_A(t)$ in $|t| \leqq q$ is a *multiple of* k: $m = lk$.

We now consider a G-transformation applied to system $\mathcal{A}$ of (2.9). Let $\{H_n(t)\}$ be a sequence of functions, analytic in $|t| \leqq q$, satisfying the condition

$$H_{nk+j}(t) = H_j(t) \qquad (j = 0, 1, \cdots, k-1; n = 0, 1, \cdots), \tag{2.15}$$

and such that

$$H_n(t) = P(t)Q_n(t)$$

where $Q_n(t)$ is analytic in $|t| \leqq q$. Suppose the order of indices $0, 1, \cdots, k-1$ in (2.11) is permuted cyclically to $r, r+1, \cdots, r+k-1$ (mod k), obtaining a determinant $\Delta_A^{(r)}(t)$ ($r=0$ corresponding to $\Delta_A(t)$ itself). Then

$$\Delta_A^{(r)}(t) = \Delta_A(t),$$

as is readily shown(5).

In $\Delta_A^{(r)}(t)$, replace the sth column ($s=0, 1, \cdots, k-1$) by $H_r(t)$, $H_r(\omega t), \cdots, H_r(\omega^{k-1}t)$, and denote the resulting determinant by $\Delta_{r,s}(t; H_r)$. Now define the functions $\lambda_{r,s}(t)$ as follows:

$$\lambda_{r,s}(t) = \frac{\Delta_{r,s}(t; H_r)}{\Delta_A(t)} \qquad (r, s = 0, 1, \cdots, k-1); \tag{2.16}$$

$$\lambda_{nk+j,s}(t) = \lambda_{j,s}(t) \qquad (j, s = 0, 1, \cdots, k-1; n = 0, 1, 2, \cdots). \tag{2.17}$$

From the last property assumed for $\{H_n(t)\}$ it is clear that the zeros of $\Delta_A(t)$ are cancelled out by each numerator, so that the functions $\{\lambda_{j,s}(t)\}$ are analytic in $|t| \leqq q$. If we then define $\{G_n(t)\}$ by the relations

$$G_n(t) = \sum_{j=0}^{k-1} \lambda_{n,j}(t), \tag{2.18}$$

we see that $G_n(t)$ is also analytic in $|t| \leqq q$, and that

$$G_{nk+j}(t) = G_j(t) \qquad (j = 0, 1, \cdots, k-1; n = 0, 1, 2, \cdots). \tag{2.19}$$

The sequences $\{A_n(t)\}$, $\{G_n(t)\}$ fulfill the conditions (2.7) of Theorem 2.1, so $\{G_n(t)\}$ determines a G-transformation on system $\mathcal{A}$. We now further specialize $\{H_n(t)\}$ by choosing $Q_n(t) \equiv 1$, so that

$$H_n(t) = P(t) \qquad (n = 0, 1, \cdots), \tag{2.20}$$

and we examine the corresponding G-transformation.

THEOREM 2.3. *Given the k-periodic system* (2.9), *where* $A_j(t)$ ($j=0, 1, \cdots, k-1$) *is analytic in* $|t| \leqq q$, *with* $A_j(0) \neq 0$, *and with* $((c_n)) \leqq q$. *Let* $\{H_n(t)\}$ *be defined by* (2.20), *and* $\{G_n(t)\}$ *by* (2.18), *so that* (2.19) *holds. Then the hypotheses of Theorem* 2.1 *are satisfied, and the resulting G-transform of* (2.9) *is the one-periodic system*

(5) For: Multiply the respective rows of $\Delta_A^{(r)}$ by $1, \omega^r, \omega^{2r}, \cdots, \omega^{(k-1)r}$ and interchange columns until the natural order of indices is restored. It is then seen, on cancelling a common factor ± 1 on both sides, that $\Delta_A^{(r)}(t) = \Delta_A(t)$.

$$\mathcal{P}: \quad p_0x_n + p_kx_{n+k} + \cdots + p_{lk}x_{n+lk} = c_n' \qquad (n = 0, 1, \cdots), \tag{2.21}$$

where

$$P(t) = p_0 + p_kt^k + \cdots + p_{lk}t^{lk} \qquad (p_{lk} = 1),$$

and where $\{c_n'\}$ *is given by* (2.3). *Every solution* $\{x_n\}$ *of* (2.9) *for which* $((x_n)) \leqq q$ *is also a solution of* (2.21).

We first observe that for each fixed r the functions $\lambda_{r,s}(t)$ of (2.16) are the unique solutions (as given by Cramer's rule) of the system of linear equations

$$P(t) = H_r(\omega^f t) = \sum_{j=0}^{k-1} \omega^{fj}\lambda_{r,j}(t)A_{r+j}(\omega^f t) \qquad (f = 0, 1, \cdots, k-1). \tag{2.22}$$

For $f=0$ we have

$$P(t) = H_r(t) = \sum_{j=0}^{k-1} \lambda_{r,j}(t)A_{r+j}(t); \tag{2.23}$$

and replacing t by $\omega^f t$ in (2.23), we obtain

$$P(t) = \sum_{j=0}^{k-1} \lambda_{r,j}(\omega^f t)A_{r+j}(\omega^f t). \tag{2.24}$$

Comparison of (2.22) and (2.24) shows that the functions $\lambda_{r,j}^*(t) = \omega^{-fj}\lambda_{r,j}(\omega^f t)$ form a solution of (2.22); and from the uniqueness of the solution of system (2.22) we conclude that

$$\lambda_{r,j}(\omega^f t) = \omega^{fj}\lambda_{r,j}(t) \qquad (f, j = 0, 1, \cdots, k-1; r = 0, 1, \cdots). \tag{2.25}$$

Thus, if $G_n(t) = \sum_{j=0}^{\infty} g_{nj}t^j$, we see from (2.18) that

$$\lambda_{r,j}(t) = g_{rj}t^j + g_{r,j+k}t^{j+k} + g_{r,j+2k}t^{j+2k} + \cdots \qquad (j = 0, 1, \cdots, k-1; r = 0, 1, 2, \cdots). \tag{2.26}$$

It follows from (2.23) that

$$P(t) = \sum_{j=0}^{\infty} g_{nj}t^jA_{n+j}(t) \qquad (n = 0, 1, \cdots). \tag{2.27}$$

Since Theorem 2.1 is applicable, we see from (2.4) that the sequence $\{H_n(t)\}$ *of that theorem* is given by $H_n(t) = P(t)$, which is also the value of $H_n(t)$ as defined by (2.20). That is, the two sequences $\{H_n(t)\}$ are identical. Consequently from Theorem 2.1 the G-transformation carries system $\mathcal{A}$ into the system $\mathcal{K} = \mathcal{A}_G$ given by (2.2); and this system $\mathcal{K}$ is precisely the one-periodic system $\mathcal{P}$ of (2.21). Thus the theorem is established.

Since the zeros of $P(t)$ come in nests of k each, we can write

$$P(t) = \prod (t - \omega^f\alpha_r)^{p_r} \qquad (f = 0, 1, \cdots, k-1; r = 1, 2, \cdots, h), \tag{2.28}$$

where $p_1+\cdots+p_h=l$ (the total number of zeros being $m=lk$), and where $\alpha_j\neq\alpha_s$ for $j\neq s$.

Consider the homogeneous case of (2.9): $c_n\equiv 0$. Then $c_n'\equiv 0$, and the corresponding system (2.21) is known to have the general solution

$$(2.29)\qquad x_n=\sum_{r=1}^{h}\sum_{f=0}^{k-1}\{A_{r,f,0}+A_{r,f,1}n+\cdots+A_{r,f,p_r-1}n^{p_r-1}\}(\omega^f\alpha_r)^n,$$

where the A's are arbitrary constants. As for the nonhomogeneous case, if $((c_n))\leqq q$ then also $((c_n'))\leqq q$ (by Theorem 2.1). On defining $C_1(t)$ by

$$(2.30)\qquad C_1(t)=\sum_{0}^{\infty}c_n't^n,$$

it is readily shown (by direct substitution) that a particular solution of (2.21) is given by

$$(2.31)\qquad x_n=\frac{1}{2\pi i}\int_\Gamma\frac{C_1(1/t)t^{n-1}}{P(t)}\,dt,$$

where Γ is the circle $|t|=R$, chosen so that $R>q$. Moreover, $((x_n))\leqq q$, since R can be chosen arbitrarily close to q. We therefore have the following theorem.

THEOREM 2.4. *Let the hypotheses of Theorem* 2.3 *be fulfilled. If* $\{x_n\}$, *of type not exceeding* q, *satisfies the homogeneous system* (2.9) $[c_n\equiv 0]$, *then there is a choice of the constants* $A_{r,f,s}$ *for which* $\{x_n\}$ *is given by* (2.29); *and if* $\{x_n\}$ *satisfies the nonhomogeneous system* (2.9), *with* $((c_n))\leqq q$, *then constants* $A_{r,f,s}$ *exist such that* $\{x_n\}$ *is the sum of* (2.29) *and* (2.31).

Consider again the homogeneous case of (2.9). One method seeking to determine the number of linearly independent solutions is to substitute (2.29) into (2.9), to see how many of the coefficients $A_{r,f,s}$ are independent. In the general case this leads to algebraic complications that can be circumvented by the method of §3. We therefore merely state here (without proof) that in the process of substituting (2.29) into (2.9), it is permitted to wipe out the summation in r, giving to r a fixed value. For each fixed r there will be a certain number N_r of independent constants; and the total number of independent solutions of (2.9) is precisely $N_1+\cdots+N_h$.

We conclude this section with the consideration of a simple but important case of (2.9), that we need for later purposes. It has already been noted that the number m of zeros of $\Delta_A(t)$ in $|t|\leqq q$ is a multiple of k: $m=lk$, multiple zeros being counted multiply.

DEFINITION. The integer l is *the order of* $\Delta_A(t)$ and also *the order of system* $\mathcal{A}$, in $|t|\leqq q$.

THEOREM 2.5. *Let* (6) $\Delta_A(t)$ *be of order zero in* $|t| \leqq q$. *Then the k-periodic system* (2.9) [*homogeneous equation*] *has no solution of type not exceeding q* (*save* $\{x_n \equiv 0\}$); *and if* $((c_n)) \leqq q$, *the nonhomogeneous system has a unique solution for which* $((x_n)) \leqq q$, *and this solution is given by*

$$(2.32) \qquad x_n = c_n' = \sum_{j=0}^{\infty} g_{nj} c_{n+j} \qquad (n = 0, 1, \cdots).$$

In the present case, $P(t) \equiv 1$, so (2.21) reduces to

$$x_n = c_n', \qquad n = 0, 1, \cdots .$$

If $c_n \equiv 0$ then $c_n' \equiv 0$, so $x_n \equiv 0$. In the nonhomogeneous case, (2.32) will be shown to be a solution of (2.9) by establishing that there is a valid G-transformation carrying (2.21) back to the original system (2.9).

To this end, regard (2.21) as a k-periodic system, and let us use the notation introduced earlier, with an asterisk to distinguish the present discussion from the earlier. To begin with, we have $A_n^*(t) = H_n(t) \equiv 1$; and we wish to show that we may take $H_n^*(t) = A_n(t)$. On making this identification, we are to have(7)

$$\begin{aligned} A_{nk}(t) &= \lambda^*_{nk,0}(t) + \cdots + \lambda^*_{nk,k-1}(t) \\ &\cdots\cdots\cdots\cdots \\ A_{nk+k-1}(t) &= \lambda^*_{nk+k-1,0}(t) + \cdots + \lambda^*_{nk+k-1,k-1}(t); \end{aligned}$$

so from (2.16),

$$\lambda^*_{rs}(t) = \frac{\Delta^*_{rs}(t; A_r)}{\Delta^*_H(t)},$$

where $\Delta^*_H(t)$ has the *constant* value $\Delta^*_H(t) = \Omega_k$, given by

$$(2.33) \qquad \Omega_k \equiv \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{k-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & \omega^{k-1} & \omega^{2(k-1)} & \cdots & \omega^{(k-1)(k-1)} \end{vmatrix}.$$

Now $\Omega_k \neq 0$, since it is a Vandermond determinant in $1, \omega, \cdots, \omega^{k-1}$. Hence the functions $\lambda^*_{rs}(t)$, and with them $\{G_n^*(t)\}$, are analytic in $|t| \leqq q$, so the G^*-transformation is a valid one. Moreover, as seen from the manner in which $\{G_n^*\}$ is constructed, this transformation carries (2.21) into (2.9); so (2.9) and (2.21) are equivalent for all sequences $\{x_n\}$ for which $((x_n)) \leqq q$.

(6) It is assumed without further mention (now and hereafter except where otherwise stated) that $A_0(t), \cdots, A_{k-1}(t)$ are analytic in $|t| \leqq q$, and that $A_j(0) \neq 0, j = 0, 1, \cdots, k-1$.

(7) Compare (2.23), using $H_r(t)$ rather than the specialized $P(t)$.

Hence (2.9) has the solution $x_n = c_n'$ given by (2.32). This completes the proof of Theorem 2.5.

3. **Factorization of k-periodic systems.** Let $\mathcal{B}$, $\mathcal{C}$ be two systems of linear forms in $\{x_n\}$:

$$(3.1)\qquad \mathcal{B}:\ \mathcal{B}_n[X] \equiv \sum_{j=0}^{\infty} \beta_{nj}x_j; \qquad \mathcal{C}:\ \mathcal{C}_n[X] \equiv \sum_{j=0}^{\infty} \gamma_{nj}x_j \qquad (n = 0, 1, \cdots),$$

where X stands for the sequence $\{x_n\}$. Suppose $\mathcal{C}$ has the property[8] that for every X of type not exceeding λ, the forms $\mathcal{C}_n[X]$ converge and define a sequence of type not exceeding μ; and let $\mathcal{B}$ be such that when $((x_n)) \leqq \mu$, then the $\mathcal{B}_n[X]$ series converge and form a sequence of type less than or equal to ν. Then when $((x_n)) \leqq \lambda$, if operation $\mathcal{C}$ is performed, and is followed by $\mathcal{B}$, there results a sequence of type not exceeding ν. Denote this combined operation by $\mathcal{A}$: $\mathcal{A} = \mathcal{B}\mathcal{C}$.

We can represent $\mathcal{A}$ (at least formally) as a system of linear forms in $\{x_n\}$:

$$(3.2)\qquad \mathcal{A}:\ \mathcal{A}_n[X] \equiv \sum_{j=0}^{\infty} \alpha_{nj}x_j \qquad (n = 0, 1, \cdots).$$

In fact, the coefficients $\{\alpha_{nj}\}$ can be obtained as follows: Define $\mathcal{A}_n(t)$, $\mathcal{B}_n(t)$, $\mathcal{C}_n(t)$ by

$$(3.3)\qquad \begin{aligned} \mathcal{A}_n(t) &= \sum_{j=0}^{\infty} \alpha_{nj}t^j, \qquad \mathcal{B}_n(t) = \sum_{j=0}^{\infty} \beta_{nj}t^j, \\ \mathcal{C}_n(t) &= \sum_{j=0}^{\infty} \gamma_{nj}t^j \qquad (n = 0, 1, \cdots). \end{aligned}$$

Then (§3 of paper in footnote 8)

$$(3.4)\qquad \alpha_{nj} = \beta_{n0}\gamma_{0j} + \beta_{n1}\gamma_{1j} + \beta_{n2}\gamma_{2j} + \cdots \qquad (n, j = 0, 1, \cdots),$$

and

$$(3.5)\qquad \mathcal{A}_n(t) = \beta_{n0}\mathcal{C}_0(t) + \beta_{n1}\mathcal{C}_1(t) + \beta_{n2}\mathcal{C}_2(t) + \cdots \qquad (n = 0, 1, \cdots).$$

It has moreover been shown (§3, paper of footnote 8) that system $\mathcal{A}$ as defined by (3.4) or (3.5) carries every $\{x_n\}$ for which $((x_n)) \leqq \lambda$ into a sequence of type not greater than ν; and that system $\mathcal{A}$ is actually (that is, not merely formally) identical with the product $\mathcal{B}\mathcal{C}$.

LEMMA 3.1. *Let systems $\mathcal{B}$, $\mathcal{C}$ of* (3.1) *each carry every $\{x_n\}$ for which $((x_n)) \leqq \lambda$ into a sequence of type not exceeding λ. Let the systems of homogeneous equations*

(8) Necessary and sufficient conditions for this are found in §2 of I. M. Sheffer, *Systems of linear equations of analytic type*, Duke Math. J. vol. 11 (1944) pp. 167–180.

$$\mathcal{B}:\quad \mathcal{B}_n[X] = 0;\qquad \mathcal{C}:\quad \mathcal{C}_n[X] = 0 \qquad\qquad (n = 0, 1, \cdots)$$

have precisely b and c linearly independent solutions respectively of type not exceeding λ; *and let each of the nonhomogeneous systems*

$$\mathcal{B}:\quad \mathcal{B}_n[X] = c_n;\qquad \mathcal{C}:\quad \mathcal{C}_n[X] = c_n \qquad\qquad (n = 0, 1, \cdots)$$

possess a solution $\{x_n\}$, *for which* $((x_n)) \leqq \lambda$, *for every sequence* $\{c_n\}$ *satisfying* $((c_n)) \leqq \lambda$. *Then the system* $\mathcal{A} = \mathcal{B}\mathcal{C}$ *has the following properties*:

(i) *Homogeneous system*

$$\mathcal{A}:\quad \mathcal{A}_n[X] = 0$$

has precisely $a = b + c$ *linearly independent solutions for which* $((x_n)) \leqq \lambda$;

(ii) *Nonhomogeneous system*

$$\mathcal{A}:\quad \mathcal{A}_n[X] = c_n$$

has a solution with $((x_n)) \leqq \lambda$ *for every* $\{c_n\}$ *such that* $((c_n)) \leqq \lambda$.

As remarked earlier in this section, the series $\mathcal{A}_n[X]$ converge and define a sequence of type not greater than λ for every $\{x_n\}$ for which $((x_n)) \leqq \lambda$. Now

$$\text{(a)}\qquad \mathcal{A}_n[X] \equiv \mathcal{B}_n[\mathcal{C}[X]].$$

Let $\{x_n^{(1)}\}, \cdots, \{x_n^{(b)}\}$ be a fundamental set of solutions of $\mathcal{B}_n[X] = 0$, and $\{y_n^{(1)}\}, \cdots, \{y_n^{(c)}\}$ a corresponding set for $\mathcal{C}_n[X] = 0$. Then we see from (a) that $X = \{x_n\}$ satisfies $\mathcal{A}_n[X] = 0$ if and only if one of the following relations holds:

$$\text{(i)}\qquad \mathcal{C}_n[X] = 0;$$

$$\text{(ii)}\qquad \mathcal{C}_n[X] = \sigma_1 x_n^{(1)} + \cdots + \sigma_b x_n^{(b)}$$

for some choice of the constants σ_j.

If (i) holds, then $\{x_n\}$ is a linear combination of $\{y_n^{(1)}\}, \cdots, \{y_n^{(c)}\}$. If (ii) holds, let $\{u_n^{(j)}\}$ be, for each $j = 1, \cdots, b$, a particular solution of $\mathcal{C}_n[\{u_r^{(j)}\}] = x_n^{(j)}$. Such solutions exist by hypothesis. The general solution of (ii) is then given by

$$\text{(b)}\qquad x_n = \sum_{j=1}^{b} \sigma_j u_n^{(j)} + \sum_{s=1}^{c} \delta_s y_n^{(s)},$$

for arbitrary δ's. The solutions $\{x_n\}$ found for case (i) are included in (b), namely by taking all the σ's zero.

Thus, if $\{x_n\}$ is a solution of $\mathcal{A}_n[X] = 0$ it has the form (b). Moreover, (b) does give a solution of $\mathcal{A}_n[X] = 0$ for arbitrary choice of the σ's and δ's, as is readily verified. Hence the system $\mathcal{A}_n[X] = 0$ has precisely $b + c$ linearly independent solutions, since the $b + c$ sequences $\{u_n^{(j)}\}$, $\{y_n^{(s)}\}$ are linearly independent. To see this, suppose σ's and δ's (not all zero) exist so that $x_n \equiv 0$. Then $\mathcal{C}_n[X]=0$. But $\mathcal{C}_n[\{\sum_{s=1}^{c} \delta_s y_n^{(s)}\}]=0$ by definition, and $\mathcal{C}_n[\{\sum_{j=1}^{b} \sigma_j u_n^{(j)}\}] = \sum \sigma_j x_n^{(j)}$; hence $\sum \sigma_j x_n^j = 0$ for all n, and since the $\{x_n^{(j)}\}$'s are independent, all σ's are zero. This also makes the δ's zero, so we have a contradiction to the assumption of linear dependence.

As for the nonhomogeneous case, let $V=\{v_n\}$ satisfy $\mathcal{B}_n[V]=c_n$, and choose $\{x_n\}$ so that $\mathcal{C}_n[X]=v_n$. Then $\mathcal{B}_n[\mathcal{C}[X]]=\mathcal{B}_n[V]=c_n$, so $\{x_n\}$ satisfies $\mathcal{A}_n[X]=c_n$, as was to be shown.

To apply the properties of products of systems to the case of k-periodic systems, we need to restrict the systems $\mathcal{B}$, $\mathcal{C}$ as follows: Choose

$$\beta_{nj} = \gamma_{nj} = 0 \qquad (j = 0, 1, \cdots, n-1; n = 1, 2, \cdots).$$

Then from (3.4) we deduce that also

$$\alpha_{nj} = 0 \qquad (j = 0, 1, \cdots, n-1).$$

We may then write

$$\mathcal{A}_n(t) = t^n A_n(t), \qquad \mathcal{B}_n(t) = t^n B_n(t), \qquad \mathcal{C}_n(t) = t^n C_n(t),$$

where (on making a change of lettering of coefficients),

$$A_n(t) = \sum_{j=0}^{\infty} a_{nj} t^j, \qquad B_n(t) = \sum_{j=0}^{\infty} b_{nj} t^j, \qquad C_n(t) = \sum_{j=0}^{\infty} c_{nj} t^j. \tag{3.6}$$

We still denote the corresponding systems of forms by $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, so that

$$\begin{aligned} &\mathcal{A}: \ A_n[X] \equiv \sum_{j=0}^{\infty} a_{nj} x_{n+j}; \qquad \mathcal{B}: \ B_n[X] \equiv \sum_{j=0}^{\infty} b_{nj} x_{n+j}; \\ &\mathcal{C}: \ C_n[X] \equiv \sum_{j=0}^{\infty} c_{nj} x_{n+j} \end{aligned} \tag{3.7}$$

$(n=0, 1, \cdots)$. Relations (3.4), (3.5) become

$$a_{nj} = b_{n0} c_{n,j} + b_{n1} c_{n+1,j-1} + \cdots + b_{nj} c_{n+j,0} \qquad (n, j = 0, 1, \cdots), \tag{3.8}$$

$$A_n(t) = b_{n0} C_n(t) + b_{n1} t C_{n+1}(t) + b_{n2} t^2 C_{n+2}(t) + \cdots \qquad (n = 0, 1, \cdots). \tag{3.9}$$

DEFINITION. Let $\{F_n(t)\}$ be a sequence of functions defining a system $\mathcal{F}$. To avoid the necessity of frequent repetition, we say that $\mathcal{F}$ *is admissible* if each function $F_n(t)$ is analytic in $|t| \leq q$, if $F_n(0) \neq 0$, $n=0, 1, \cdots$, and if $\mathcal{F}$ is k-periodic (which is to say, $F_{j+nk}(t) = F_j(t)$, $j=0, 1, \cdots, k-1$; $n=0, 1, \cdots$). We also say that *a sequence* $X = \{x_n\}$ *is admissible* if $((x_n)) \leq q$. The number q is to be positive.

Now let $\mathcal{A}$ be an admissible system. We pose the question: Do admissible k-periodic systems $\mathcal{B}$, $\mathcal{C}$ exist such that

$$\mathcal{A} = \mathcal{B}\mathcal{C}? \tag{3.10}$$

We shall see that the answer is in the affirmative.

THEOREM 3.1. *Let $\mathcal{B}$ and $\mathcal{C}$ be admissible and define $\mathcal{A}$ by $\mathcal{A}=\mathcal{B}\mathcal{C}$. Then $\mathcal{A}$ is admissible, and*

$$\Delta_A(t) = \Omega'_k\Delta_B(t)\Delta_C(t), \tag{3.11}$$

where Ω'_k is the nonzero constant

$$\Omega_k^{1} = \frac{1}{k^k}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & (1/\omega) & \cdots & (1/\omega)^{k-1} \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ 1 & (1/\omega)^{k-1} & \cdots & (1/\omega)^{(k-1)(k-1)} \end{vmatrix}. \tag{3.12}$$

Since $\mathcal{B}$, $\mathcal{C}$ are k-periodic,

$$B_{nk+j}(t) = B_j(t), \qquad C_{nk+j}(t) = C_j(t) \tag{3.13}$$
$$(j = 0, 1, \cdots, k-1;\ n = 0, 1, \cdots);$$

so from (3.9),

$$A_n(t) = \sum_{j=0}^{k-1} C_{n+j}(t)\left\{\sum_{r=0}^{\infty} b_{n,j+rk}t^{j+rk}\right\} \qquad (n = 0, 1, \cdots). \tag{3.14}$$

This shows that $\mathcal{A}$ is also k-periodic.

Now $\omega=e^{2\pi i/k}$, $\omega^k=1$, so

$$\sum_{m=0}^{k-1}(\omega^s)^m = \begin{cases} k & \text{if } s \equiv 0 \pmod{k}, \\ 0 & \text{if } s \not\equiv 0 \pmod{k}. \end{cases} \tag{3.15}$$

Consequently, for $j=0, 1, \cdots, k-1$; $s=0, 1, \cdots, k-1$,

$$\frac{1}{k}\sum_{p=0}^{k-1}\omega^{k-ps}B_j(\omega^p t) = \sum_{r=0}^{\infty} b_{j,s+rk}t^{s+rk}; \tag{3.16}$$

so (3.14) becomes

$$A_p(t) = \frac{1}{k}\sum_{j=0}^{k-1} C_{j+p}(t)\left\{\sum_{r=0}^{k-1}\omega^{k-rj}B_p(\omega^r t)\right\} \qquad (p = 0, 1, \cdots, k-1). \tag{3.17}$$

If the values of $A_p(t)$, $A_p(\omega t)$, $\cdots$ as obtained from (3.17) are substituted into (2.21) it is seen by a straightforward computation that $\Delta_A(t)$ is the product of two determinants, the first of which is $\Delta_C(t)$. The other is $(1/k^k)|h_{jr}|$, where

$$h_{jr} = \sum_{s=0}^{k-1}\omega^{k-(j-r)s}B_r(\omega^s t) \qquad (j, r = 0, 1, \cdots, k-1).$$

Determinant $(1/k^k)|h_{jr}|$ also factors into two, namely

$$\frac{1}{k^k}|h_{jr}| = \Omega_k' \cdot \Delta_B(t),$$

as is easily found. That $\Omega_k' \neq 0$ follows from the fact that it is a Vandermond determinant in $1, \omega^{-1}, \cdots, \omega^{-(k-1)}$.

COROLLARY 3.1. *Ω_k' and Ω_k (of 2.33) are reciprocals:*

$$\Omega_k' = \frac{1}{\Omega_k} \cdot \tag{3.18}$$

For, since Ω_k' is independent of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ we may choose $\mathcal{B}$ and $\mathcal{C}$ to be the identity system: $B_n(t) = C_n(t) = 1$, $n = 0, 1, \cdots$. Then $\mathcal{A}$ is also the identity, and

$$\Delta_A(t) = \Delta_B(t) = \Delta_C(t) = \Omega_k;$$

so (3.18) holds.

REMARK. It is not necessary that $\mathcal{B}$ and $\mathcal{C}$ be admissible in order that (3.11) hold. For example, if $\mathcal{B}$, $\mathcal{C}$ are k-periodic and $\{B_n(t)\}$, $\{C_n(t)\}$ are analytic in $|t| \leqq q$, then $\mathcal{A}$ has the same property, and (3.11) is valid. More generally, $\mathcal{B}$ and $\mathcal{C}$ may be general (and hence formal) k-periodic systems defined by the (formal) series (3.7). Then $\mathcal{A}$, defined by (3.14), will also be a formal k-periodic system, and (3.11) will hold formally.

As a simple extension of Theorem 3.1 we have the following theorem.

THEOREM 3.2. *Let $\mathcal{B}_1, \cdots, \mathcal{B}_p$ be admissible and let $\mathcal{A} = \mathcal{B}_1\mathcal{B}_2 \cdots \mathcal{B}_p$. Then $\mathcal{A}$ is admissible and*

$$\Delta_A(t) = \Omega_k'^{p-1}\Delta_{B_1}(t)\Delta_{B_2}(t) \cdots \Delta_{B_p}(t). \tag{3.19}$$

REMARK. The preceding remark applies here also.

LEMMA 3.2. *Let $\mathcal{B}$ be admissible, and let $\Delta_B(t)$ be expanded according to the elements of the jth column $(j = 0, 1, \cdots, k-1)$:*

$$\Delta_B(t) = \sum_{r=0}^{k-1} \omega^{rj}B_j(\omega^r t)\Delta_{rj}(t) \equiv \sum_{r=0}^{k-1} K_{rj}(t), \tag{3.20}$$

where the functions $\Delta_{rj}(t)$ are cofactors and

$$K_{rj}(t) = \omega^{rj}B_j(\omega^r t)\Delta_{rj}(t). \tag{3.21}$$

Then for $r, j = 0, 1, \cdots, k-1$,

$$K_{rj}(t) = K_{0j}(\omega^r t). \tag{3.22}$$

To see this observe that

$$K_{rj}^* \equiv \frac{K_{rj}(t)\omega^{-rj}(-1)^{j+r}}{B_j(\omega^r t)}$$

is the minor obtained by striking out column j and row r from $\Delta_B(t)$. (It should be borne in mind that j and r run from 0 to $k-1$ rather than from 1 to k.) In the determinant for $K_{0j}^*(t)$, multiply each column by that power of ω given by r times the subscript on the corresponding B's of that column. Here r has a *fixed* value from 1 to $k-1$. This multiplies $K_{0j}^*(t)$ by ω^u where $u = r\sum_{s=0}^{k-1} s - rj$. Since $\omega = e^{2\pi i/k}$, one finds that $\omega^u = \omega^{-rj}(-1)^{r(k-1)}$.

In this new determinant, whose value is $\omega^{-rj}(-1)^{r(k-1)}K_{0j}^*(t)$, replace t by $\omega^r t$. Then in the sth row, each B has the argument $\omega^{r+s+1}t$ ($s = 0, 1, \cdots, k-2$), so in the $(k-r-1)$th row the argument of the B's is t. Suppose this row is moved to the top ($k-r-1$ moves), then the $(k-r)$th row is moved to the second row (also $k-r-1$ moves), and so on. The total number of moves made is $r(k-r-1)$, so the determinant has been multiplied by $(-1)^{r(k-r-1)}$. The resulting determinant is now seen to be precisely the minor obtained from $\Delta_B(t)$ by crossing out the jth column and rth row. In other words,

$$\omega^{-rj}(-1)^{r(k-1)}K_{0j}^*(\omega^r t) = (-1)^{r(k-r-1)}K_{rj}^*(t).$$

From this and the definition of K_{rj}^* one obtains (3.22).

COROLLARY 3.2. *For each* $j = 0, 1, \cdots, k-1$,

$$\Delta_B(t) = \sum_{r=0}^{k-1} K_j(\omega^r t), \tag{3.23}$$

where $K_j(t) \equiv K_{0j}(t)$.

REMARK. The remark following Corollary 3.1 applies to Lemma 3.2 and to Corollary 3.2.

We can now take up the problem of factoring the admissible system $\mathcal{A}$. To accomplish this we must establish the existence of $\mathcal{B}$ and $\mathcal{C}$ with the following properties:

(i) $\mathcal{B}$ and $\mathcal{C}$ are k-periodic.
(ii) $\{B_n(t)\}$, $\{C_n(t)\}$ are analytic in $|t| \leq q$.
(iii) $B_n(0) \neq 0$, $C_n(0) \neq 0$, $n = 0, 1, \cdots$.
(iv) $\mathcal{A} = \mathcal{B}\mathcal{C}$.

Now condition (i) we satisfy by edict; that is, we shall only consider sets $\{B_n(t)\}$, $\{C_n(t)\}$ that are k-periodic:

$$B_{j+nk}(t) = B_j(t); \qquad C_{j+nk}(t) = C_j(t) \qquad (j = 0, 1, \cdots, k-1).$$

Condition (iv) will then hold if equations (3.17) are satisfied. We shall in fact *determine* $\mathcal{B}$, $\mathcal{C}$ by means of (3.17) and a further condition soon to be stated, and shall then show that conditions (ii) and (iii) are fulfilled.

First we dispose of the simple case that $\mathcal{A}$ is of order *zero* in $|t| \leqq q$. The factorization is then $\mathcal{A} = \mathcal{B}\mathcal{C}$ where $\mathcal{B}$ is the *identity* system $B_n(t) \equiv 1$, and $\mathcal{C} \equiv \mathcal{A}$.

Suppose then that $\mathcal{A}$ is of positive order in $|t| \leqq q$, and let $\{\omega^f \alpha\}$, $f = 0, 1, \cdots, k-1$, be a nest of zeros of $\Delta_A(t)$. We impose on $\mathcal{B}$ the condition

$$\Delta_B(t) = -\alpha^k + t^k, \tag{3.24}$$

which condition is consistent with the fact of Theorem 2.2 that the power series for $\Delta_B(t)$ contains only powers of t^k.

Consider the system of equations

$$\sum_{j=0}^{k-1} d_{ij} u_j = 0 \qquad (i = 0, 1, \cdots, k-1) \tag{3.25}$$

where (d_{ij}) is the matrix obtained from $\Delta_A(t)$ by choosing $t = \alpha$. Since $\Delta_A(\alpha) = 0$, the determinant of (3.25) is zero, so a solution $u_0, \cdots, u_{k-1}$ exists for which not all u's are zero. Let s be chosen, *and fixed from now on*, so that $u_s \neq 0$.

Taking $j = s$ in (3.23),

$$\Delta_B(t) = \sum_{r=0}^{k-1} K_s(\omega^r t),$$

so that by (3.24),

$$-\alpha^k + t^k = \sum_{r=0}^{k-1} K_s(\omega^r t). \tag{3.26}$$

Set $K_s(t) = \sum_{j=0}^{\infty} d_j t^j$. From (3.26) it is found that

$$d_0 = -\frac{1}{k}\alpha^k, \qquad d_k = \frac{1}{k}, \qquad d_{rk} = 0 \qquad (r = 2, 3, \cdots),$$

while all other d's are arbitrary. The most general power series $\sum d_j t^j$ satisfying (3.26) therefore has the form

$$K_s(t) = \sum_{j=0}^{k-1} t^j E_j(t),$$

where $E_0(t) = (1/k)(-\alpha^k + t^k)$ and $E_1, \cdots, E_{k-1}$ are arbitrary power series containing only powers of t^k. Choose $E_j(t) \equiv \lambda_j$ $(j = 1, \cdots, k-1)$, where the λ's are constants (as yet undetermined). Then a solution of (3.26) is given by

$$K_s(t) = \frac{1}{k}(-\alpha^k + t^k) + \sum_{j=1}^{k-1} \lambda_j t^j. \tag{3.27}$$

This function is of course analytic in $|t| \leqq q$.

We pass now to the determination of the functions $\{B_n(t)\}$. Each of

$B_0, \cdots, B_{k-1}$ *save* B_s we chóose to be identically one:

$$B_j(t) = 1 \qquad (j \neq s,\ 0 \leqq j \leqq k-1). \tag{3.28}$$

As for $B_s(t)$, we choose it so that (3.24) holds. From its definition,

$$K_s(t) = K_{0s}(t) = B_s(t)\Delta_{0s}(t)$$

$$= (-1)^s B_s(t) \begin{vmatrix} B_0(\omega t) \cdots \omega^{s-1}B_{s-1}(\omega t) & \omega^{s+1}B_{s+1}(\omega t) \cdots \omega^{k-1}B_{k-1}(\omega t) \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ B_0(\omega^{k-1}t) \cdots \omega^{(k-1)(s-1)}B_{s-1}(\omega^{k-1}t) & \omega^{(k-1)(s+1)}B_{s+1}(\omega^{k-1}t) \cdots \omega^{(k-1)(k-1)}B_{k-1}(\omega^{k-1}t) \end{vmatrix};$$

so from (3.28),

$$K_s(t) = \Omega_{k,s}B_s(t), \tag{3.29}$$

where

$$\Omega_{k,s} = (-1)^s \begin{vmatrix} 1 & \omega & \cdots & \omega^{s-1} & \omega^{s+1} & \cdots & \omega^{k-1} \\ 1 & \omega^2 & \cdots & \omega^{2(s-1)} & \omega^{2(s+1)} & \cdots & \omega^{2(k-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \omega^{k-1} & \cdots & \omega^{(k-1)(s-1)} & \omega^{(k-1)(s+1)} & \cdots & \omega^{(k-1)(k-1)} \end{vmatrix}. \tag{3.30}$$

That $\Omega_{k,s} \neq 0$ is seen from the fact that it is a power of ω multiplied by a Vandermond determinant in $1, \omega, \cdots, \omega^{s-1}, \omega^{s+1}, \cdots, \omega^{k-1}$. Consequently,

$$B_s(t) = \frac{1}{\Omega_{k,s}} K_s(t) = \frac{-\alpha^k + t^k}{k\Omega_{k,s}} + \sum_{j=1}^{k-1} \mu_j t^j \equiv \sum_{j=0}^{k} \mu_j t^j, \tag{3.31}$$

where $\mu_1, \cdots, \mu_{k-1}$ are arbitrary constants, and

$$\mu_0 = \frac{-\alpha^k}{k\Omega_{k,s}}, \qquad \mu_k = \frac{1}{k\Omega_{k,s}}. \tag{3.32}$$

We have now found a k-periodic system $\mathcal{B}$ for which (3.24) holds. Each $B_n(t)$ is analytic in $|t| \leqq q$, and

$$B_s(0) = \frac{-\alpha^k}{k\Omega_{k,s}} \neq 0,$$

so $B_j(0) \neq 0, j = 0, 1, \cdots, k-1$. Hence $\mathcal{B}$ is an admissible system.

Turning to $\mathcal{C}$, we find from (3.9) that

$$C_j(t) = A_j(t) \quad (j \neq s,\ j = 0, 1, \cdots, k-1); \tag{3.33}$$

and that

$$A_s(t) = \mu_0 C_s(t) + \mu_1 t A_{s+1}(t) + \cdots + \mu_{k-1} t^{k-1} A_{s+k-1}(t) + \mu_k t^k C_s(t)$$

(since we are to have $C_{s+k}(t) = C_s(t)$). Hence

$$C_s(t) = \frac{1}{\mu_0 + \mu_k t^k}\left\{A_s(t) - \sum_{r=1}^{k-1} \mu_r t^r A_{s+r}(t)\right\}. \tag{3.34}$$

System $\mathcal{C}$ is now defined, and is k-periodic; and it satisfies $\mathcal{A} = \mathcal{B}\mathcal{C}$. The property of admissibility remains to be examined. It is clear from (3.34) that $C_s(t)$ will not be analytic for all choices of $\{\mu_j\}$. In fact, since $\mu_0 + \mu_k t^k$ has the zeros $\omega^f \alpha$ ($f = 0, 1, \cdots, k-1$), a necessary and sufficient condition that $C_s(t)$ be analytic (in $|t| \leqq q$) is that the following system of equations be fulfilled:

$$A_s(\omega^f \alpha) - \sum_{r=1}^{k-1} A_{s+r}(\omega^f \alpha) \mu_r \omega^{fr} \alpha^r = 0 \qquad (f = 0, 1, \cdots, k-1). \tag{3.35}$$

Regarding the "unknowns" in (3.35) as $-1, \mu_1, \cdots, \mu_{k-1}$, the determinant of the coefficients is, after removing an obvious power of α and multiplying the rows by suitable powers of ω, precisely the determinant $\Delta_A(\alpha)$; at least to within a permutation of columns. As $\Delta_A(\alpha) = 0$, the above system (3.35) has a solution $z_0, z_1, \cdots, z_{k-1}$. We may therefore identify the z's with $-1, \mu_1, \cdots, \mu_{k-1}$ provided that $z_0 \neq 0$. Now except for column permutations, system (3.35) coincides with system (3.25); and the permutation is such that $z_0 = u_s$. But $u_s \neq 0$. Hence we may take $z_0 = -1$. In other words, constants, $\mu_1, \cdots, \mu_{k-1}$ exist to satisfy (3.35). We suppose from now on that $\mu_1, \cdots, \mu_{k-1}$ have been given values for which (3.35) is true. Then the functions $C_n(t)$ are all analytic in $|t| \leqq q$. Also, if $j \neq s$, $C_j(0) = A_j(0) \neq 0$; and $C_s(0) = (1/\mu_0) A_s(0) \neq 0$. Hence $\mathcal{C}$ is an admissible system.

To sum up, we have the following theorem.

THEOREM 3.3. *Let $\mathcal{A}$ be an admissible system in $|t| \leqq q$. If $\mathcal{A}$ is of order zero it has the decomposition*

$$\mathcal{A} = \mathfrak{J}\mathcal{A}$$

where $\mathfrak{J}$ is the identity system $[I_n(t) \equiv 1]$. If $\mathcal{A}$ is of positive order l, it has the factorization

$$\mathcal{A} = \mathcal{B}\mathcal{C}$$

where $\mathcal{B}$, $\mathcal{C}$ are the admissible systems defined by (3.28), (3.31), (3.33), (3.34), (3.35). *Moreover, $\mathcal{B}$, $\mathcal{C}$ are of respective orders one and $l-1$.*

All has been shown save the matter of order. That $\mathcal{B}$ is of order one is seen from its definition; and the order of $\mathcal{C}$ then follows from (3.11).

DEFINITION. The determination of system $\mathcal{B}$ involves the zeros $\{\omega^f \alpha\}$ and the index s. When it is necessary to emphasize this (as it sometimes is), we say that $\mathcal{B}$ *corresponds to the zeros* $\{\omega^j\alpha\}$ *(or simply to the zero* α*) and to the index* s.

We are now in the position to step from the zero-order case of Theorem 2.5 to that of a system of order one.

THEOREM 3.4. *Let* $\Delta_A(t)$ *be of order one in* $|t|\leqq q$, *and consider the k-periodic system* (2.9), *where the functions* $\{A_n(t)\}$ *are analytic in* $|t|\leqq q$ *and* $A_n(0)\neq 0$ $(n=0, 1, \cdots)$. *The homogeneous case* $[c_n\equiv 0]$ *has precisely one linearly independent solution* $\{x_n\}$ *of type not exceeding* q; *and if* $((c_n))\leqq q$, *the nonhomogeneous system always has a solution* $\{x_n\}$ *with* $((x_n))\leqq q$, *and the most general such solution contains one arbitrary constant.*

Consider the homogeneous system. Since $\mathcal{A}$ is of first order, we can write $\mathcal{A}=\mathcal{B}\mathcal{C}$, where $\mathcal{C}$ is of zero order, so we know from Theorem 2.5 that $\mathcal{C}[X]=0$ has $X\equiv 0$ as its only solution. Consequently, by Lemma 3.1, systems $\mathcal{A}[X]=0$, $\mathcal{B}[X]=0$ have the same number of independent solutions. We need therefore only consider $\mathcal{B}[X]=0$. When written out, this system is

$$x_n = 0 \qquad\qquad (n \not\equiv s \pmod{k}),$$

$$x_{nk+s} - \alpha^k x_{(n-1)k+s} = -k\Omega_{k,s}\sum_{j=1}^{k-1}\mu_j x_{(n-1)k+s+j} = 0;$$

so the general solution is

$$x_n = 0 \quad (n \not\equiv s); \qquad x_{nk+s} = \alpha^{nk}x_s \qquad\qquad (x_s \text{ arbitrary}).$$

Thus there is only one independent solution, and it is of type $|\alpha|$.

We turn to the nonhomogeneous system. If it is established that system

$$\mathcal{B}:\ B_n[X] = c_n \qquad\qquad (n = 0, 1, \cdots) \tag{3.36}$$

has an admissible solution $X=\{x_n\}$ for every admissible $\{c_n\}$, then $U=\{u_n\}$ will exist to satisfy

$$\mathcal{A}:\ A_n[U] = c_n \qquad\qquad (n = 0, 1, \cdots),$$

so Theorem 3.4 will have been proved. For, define U as the unique solution of

$$\mathcal{C}:\ C_n[U] = x_n \qquad\qquad (n = 0, 1, \cdots)$$

guaranteed to exist by Theorem 2.5. Then

$$A_n[U] = B_n[C[U]] = B_n[X] = c_n \qquad\qquad (n = 0, 1, \cdots).$$

It suffices therefore to show that (3.36) always has a solution.

From (3.28) and (3.31) we see that system (3.36) has the form

$$x_r = c_r \qquad\qquad (r \not\equiv s \pmod{k});$$

$$-\alpha^k x_{s+nk} + x_{s+(n+1)k} = k\Omega_{k,s}c_{s+nk} - \sum_{j=1}^{k-1}\sigma_j x_{s+nk+j} \qquad (n = 0, 1, \cdots), \tag{3.37)}$$

where (from (3.32))

$$\sigma_j = k\Omega_{k,s}\mu_j, \qquad j = 1, \cdots, k-1.$$

For $r \neq s+nk$, $x_r = c_r$; and the quantities

$$y_n \equiv x_{s+nk} \qquad (n = 0, 1, \cdots)$$

are to satisfy the system

$$(3.38) \qquad -\alpha^k y_n + y_{n+1} = \delta_n \qquad (n = 0, 1, \cdots)$$

where

$$(3.39) \qquad \delta_n = k\Omega_{k,s}c_{s+nk} - \sum_{j=1}^{k-1} \sigma_j c_{s+nk+j}.$$

It is to be noted that $((\delta_n)) \leqq q^k$ since $((c_n)) \leqq q$.

For arbitrary y_0 the general solution of (3.38) is

$$(3.40) \qquad y_n = \alpha^{nk}y_0 + \sum_{j=0}^{n-1} \alpha^{jk}\delta_{n-j-1};$$

and from this we readily determine that $((y_n)) \leqq q^k$. This fact, together with the relation $x_r = c_r$ $(r \neq s+nk)$, leads to the conclusion that $((x_n)) \leqq q$. Thus (3.36) has been shown to have an admissible solution $\{x_n\}$ for every admissible $\{c_n\}$, so the proof of Theorem 3.4 is complete.

Let us return to the admissible system $\mathcal{A}$, assumed to be of positive order l. We can write $\mathcal{A} = \mathcal{B}\mathcal{C}$ with $\mathcal{C}$ of order $l-1$. Now if $l-1>0$, $\mathcal{C}$ can likewise be factored; and so on. This gives us the following theorem.

THEOREM 3.5. *Let $\mathcal{A}$ be admissible and of positive order l in $|t| \leqq q$, with the nests of zeros*(9) $\{\omega^f\alpha_j\}$ $(j=1, \cdots, l)$. *There exists an admissible system $\mathcal{C}$ of order zero, and l admissible first order systems $\mathcal{B}_j$ $(j=1, \cdots, l)$ with $\mathcal{B}_j$ corresponding to the nest $\{\omega^f\alpha_j\}$ $(f=0, 1, \cdots, k-1)$, such that $\mathcal{A}$ has the factorization*

$$(3.41) \qquad \mathcal{A} = \mathcal{B}_1\mathcal{B}_2 \cdots \mathcal{B}_l\mathcal{C}.$$

Let $\mathcal{A}$ be of order $l>0$ and write $\mathcal{A} = \mathcal{B}\mathcal{C}$ with $\mathcal{B}$ of order one. Suppose the homogeneous system

$$\mathcal{C}: \ C_n[X] = 0 \qquad (n = 0, 1, \cdots)$$

has exactly c linearly independent admissible solutions, and that the non-homogeneous system

$$\mathcal{C}: \ C_n[X] = c_n \qquad (n = 0, 1, \cdots)$$

has an admissible solution for every admissible sequence $\{c_n\}$. Since $\mathcal{B}$ is of order one, Theorem 3.4 applies, as does also Lemma 3.1. We conclude that

(9) Since $\Delta_A(t)$ may have multiple zeros, two or more nests may have the same set of zeros.

system $\mathcal{A}$ (homogeneous case) has exactly $c+1$ linearly independent admissible solutions, and that the nonhomogeneous system $\mathcal{A}$: $A_n[X]=c_n$ has an admissible solution for every admissible $\{c_n\}$.

By induction from $\mathcal{A}=\mathcal{B}\mathcal{C}$ to (3.41) we then have the following theorem.

THEOREM 3.6. *Let $\mathcal{A}$ be admissible, of positive order l in $|t|\leqq q$. The homogeneous system*

$$\mathcal{A}:\quad A_n[X] = 0 \qquad (n = 0, 1, \cdots) \tag{3.42}$$

has precisely l linearly independent admissible solutions $\{x_n^{(j)}\}$, $j=1, \cdots, l$; and the nonhomogeneous system

$$\mathcal{A}:\quad A_n[X] = c_n \qquad (n = 0, 1, \cdots) \tag{3.43}$$

has an admissible solution $\{x_n^{(0)}\}$ for each admissible $\{c_n\}$, and the most general such solution is given by

$$x_n = x_n^{(0)} + \sum_{j=1}^{l} \delta_j x_n^{(j)} \qquad (n = 0, 1, \cdots)$$

with arbitrary constants, $\delta_1, \cdots, \delta_l$.

Theorem 3.6 gives a complete answer to the problem of solving the system of equations (1.4) defined by the general admissible system $\mathcal{A}$.

4. **Algebraic properties.** In this section we consider certain algebraic properties of systems $\mathcal{A}$, together with miscellaneous results that relate to systems. Some of these we shall need in the sections that follow, where we treat the case of k-periodic systems with perturbations.

First we restate some definitions and give some new ones. By a *system $\mathcal{A}$* we mean an infinite sequence of linear forms

$$\sum_{j=0}^{\infty} a_{nj} x_{n+j} \qquad (n = 0, 1, \cdots). \tag{4.1}$$

In other words, all we are given is the set of coefficients $\{a_{nj}\}$. A system $\mathcal{A}$ is *k-periodic* if

$$a_{nk+r,j} = a_{r,j} \qquad (r = 0, 1, \cdots, k-1; j, n = 0, 1, \cdots). \tag{4.2}$$

The *k-periodic* system $\mathcal{A}$ is *semi-admissible* if the functions

$$A_n(t) = \sum_{j=0}^{\infty} a_{nj} t^j \tag{4.3}$$

are analytic in $|t|\leqq q$; and finally, the semi-admissible system $\mathcal{A}$ is *admissible* if

$$A_j(0) \neq 0 \qquad (j = 0, 1, \cdots, k-1). \tag{4.4}$$

DEFINITION. Let Θ be the class of *all systems*, and Γ the class of all *semi-admissible systems*. Θ and Γ are obviously additive Abelian groups, and Γ is a subset of Θ.

For the *identity* we reserve the symbol $\mathfrak{J}$, and for the *zero system* $\mathcal{Q}$, so that $I_n(t)\equiv 1$ and $O_n(t)\equiv 0$ for all n.

LEMMA 4.1. *Let $\mathcal{A}\in\Theta$. Then each of the relations (separately)*

$$X\mathcal{A} = \mathfrak{J}, \qquad \mathcal{A}Y = \mathfrak{J} \tag{4.5}$$

has a solution in Θ if and only if $\mathcal{A}$ is nonsingular; that is,

$$a_{n0} \neq 0 \qquad (n = 0, 1, \cdots). \tag{4.6}$$

And when (4.6) *holds, X and Y are unique and equal, their common value being the inverse of $\mathcal{A}$:*

$$X = Y = \mathcal{A}^{-1}. \tag{4.7}$$

Moreover, $\mathcal{A}$ is also the inverse of $\mathcal{A}^{-1}$.

The lemma follows by standard algebraic argument; as does also the following lemma.

LEMMA 4.2. *If $\mathcal{A}\in\Theta$ is nonsingular, the relations*

$$X\mathcal{A} = \mathcal{A}, \qquad \mathcal{A}Y = \mathcal{A} \tag{4.8}$$

have each the unique solution $X=Y=\mathfrak{J}$.

LEMMA 4.3. *Let $\mathcal{A}\in\Gamma$. In order that each (separately) of relations* (4.5) *have a solution in Γ it is necessary and sufficient that $\mathcal{A}$ be admissible and of order zero; and in such case X and Y are unique and equal, so that* (4.7) *again holds; and $\mathcal{A}^{-1}$ is admissible of order zero, and its inverse is $\mathcal{A}$.*

(i) Suppose $X\in\Gamma$ satisfies (4.5). From (3.11) and the remark following Corollary 3.1, both $\Delta_X(t)$ and $\Delta_A(t)$ are different from zero in $|t|\leqq q$, so X and $\mathcal{A}$ are admissible and of order zero; and correspondingly for Y.

(ii) Suppose $\mathcal{A}$ is admissible of order zero. We make the observation that *when a G-transformation (of §2) is applied to a k-periodic system $\mathcal{A}$, the result is precisely the product*[10] $\mathcal{G}\mathcal{A}$. This is seen on comparing (2.4) with (3.9). In the present case, $\mathcal{A}$ is of order zero, so $P(t)\equiv 1$; thus the product $\mathcal{G}\mathcal{A}$ is precisely the identity $\mathfrak{J}$. Since $\mathcal{G}$ is admissible, as we know from §2, therefore $X=\mathcal{G}$ is an admissible solution of (4.5), and X is of order zero. Also, from Lemma 4.1, X is unique. Now

(10) That is, $\mathfrak{H}=\mathcal{G}\mathcal{A}$. Since $\mathfrak{H}$ was chosen to be one-periodic: $H_n(t)=P(t)$, we see that $\Delta_H(t)$ contains every zero of $\Delta_A(t)$ k-fold, so it is small wonder that the system of equations (2.21) defined by $\mathfrak{H}$ has an overabundance of solutions. Moreover, if $\mathcal{A}$ is of positive order, we see from $\mathfrak{H}=\mathcal{G}\mathcal{A}$ and (3.11) that $\mathcal{G}$ is not of order zero.

$$X\mathcal{A}X = \mathfrak{J}X = X,$$

and since X is in Θ and is nonsingular, there is by Lemma 4.2 a unique system U in Θ such that $XU=X$, namely $U=\mathfrak{J}$. Hence from $U=\mathcal{A}X$ we get $\mathcal{A}X=\mathfrak{J}$. Moreover by Lemma 4.1 this X is unique in Θ, so the second relation of (4.5) has the unique solution $Y=X$. Writing $\mathcal{A}^{-1}$ for this common value (which is unique since X is), we readily find that the inverse of $\mathcal{A}^{-1}$ is $\mathcal{A}$ itself.

LEMMA 4.4. *If $\mathcal{A}$ and $\mathcal{B}$ are in Θ and $\mathcal{A}$ is nonsingular, then unique systems X, Y in Θ exist such that*

$$X\mathcal{A} = \mathcal{B}, \qquad \mathcal{A}Y = \mathcal{B}. \tag{4.9}$$

In fact, it is seen that the respective solutions are

$$X = \mathcal{B}\mathcal{A}^{-1}, \qquad Y = \mathcal{A}^{-1}\mathcal{B}. \tag{4.10}$$

We have similarly the following lemma.

LEMMA 4.5. *If $\mathcal{A}$ and $\mathcal{B}$ are in Γ and $\mathcal{A}$ is admissible of order zero, then unique systems X, Y exist* (*and they are in* Γ) *such that relations* (4.9) *hold. These solutions are given by* (4.10), *and if $\mathcal{B}$ is admissible then so are X and Y.*

If $\mathcal{A}$ is of positive order and $\mathcal{B}$ is in Γ, then neither of relations (4.9) need have a solution. From (3.11) of Theorem 3.1 we can however state:

LEMMA 4.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be in Γ. A necessary condition that $X\in\Gamma$ exist such that $X\mathcal{A}=\mathcal{B}$ (or that $Y\in\Gamma$ exist with $\mathcal{A}Y=\mathcal{B}$) is that every zero of $\Delta_A(t)$ (in $|t|\leqq q$) be a zero of $\Delta_B(t)$, the multiplicity of each zero of $\Delta_A(t)$ not exceeding that of the same zero of $\Delta_B(t)$.*

From the preceding results we conclude that Θ *and* Γ *are noncommutative rings possessing a unity element* ($\mathfrak{J}$). Moreover, *$\mathcal{A}$ is a unit of Θ if and only if it is nonsingular, and is a unit of Γ if and only if it is admissible of order zero. Both* Θ and Γ have *zero divisors.*

LEMMA 4.7. *If $\mathcal{A}\neq\mathfrak{Q}$ is a zero divisor in Θ or Γ, then $\mathcal{A}$ is singular* (*that is, $a_n=0$ for at least one n*). *This condition is sufficient in Θ but not in Γ.*

(i) Suppose $\mathcal{A}\mathcal{B}=\mathfrak{Q}$ with $\mathcal{A}\neq\mathfrak{Q}$, $\mathcal{B}\neq\mathfrak{Q}$. If $\mathcal{A}$ is nonsingular, then by Lemma 4.4, $\mathcal{B}=\mathcal{A}^{-1}\mathfrak{Q}=\mathfrak{Q}$, a contradiction. So $\mathcal{A}$ must be singular.

(ii) Let $\mathcal{A}\neq\mathfrak{Q}$ in Θ be singular. There is a smallest integer r for which $a_{r0}=0$. Choose $B_r(t)=1$, $B_n(t)=0$ for $n>r$; and for $n<r$ (if $r>0$) define $B_n(t)$ by the equations

$$0 = a_{n0}B_n(t) + \cdots + a_{n,r-n-1}t^{r-n-1}B_{r-1}(t) + a_{n,r-n}t^{r-n} \quad (n = 0, 1, \cdots, r-1).$$

Then $\mathcal{B}\neq\mathfrak{Q}$ and $\mathcal{A}\mathcal{B}=\mathfrak{Q}$, so $\mathcal{A}$ is a divisor of zero.

That $\mathcal{A}\neq\mathfrak{Q}$ in Γ may be singular and yet not be a zero divisor is seen from the following example:

$$\mathcal{A}: \quad A_0(t) = t^k, \qquad A_j(t) = 1, \qquad j = 1, \cdots, k-1.$$

If $\mathcal{B}$ in Γ is chosen so that $\mathcal{A}\mathcal{B} = \mathfrak{Q}$, then

$$t^k B_0(t) = 0; \qquad B_j(t) = 0, \qquad j = 1, \cdots, k-1;$$

that is, $\mathcal{B} = \mathfrak{Q}$. Hence $\mathcal{A}$ is not a zero divisor.

On the other hand, Γ does possess zero divisors. Thus, if $\mathcal{A}$, $\mathcal{B}$ are defined by

$$\mathcal{A}: \quad A_0(t) = 1 + t, \qquad A_j(t) = 0, \qquad j = 1, 2, \cdots, k-1;$$

$$\mathcal{B}: \quad B_0(t) = t, \qquad B_1(t) = -1, \qquad B_j(t) = 0, \qquad j = 2, \cdots, k-1,$$

then $\mathcal{A}$, $\mathcal{B}$ are in Γ and $\mathcal{A}\mathcal{B} = \mathfrak{Q}$.

DEFINITION. A function $F(t)$ is a Δ-*function* if it has the following properties:

(i) $F(t) \not\equiv 0$.

(ii) $F(t)$ is analytic in $|t| \leqq q$.

(iii) The power series for $F(t)$ about $t=0$ contains only powers of t^k, so that

$$F(\omega^f t) = F(t) \qquad (f = 0, 1, \cdots).$$

Moreover, F is *singular* or *nonsingular* according as $F(0) = 0$ or $F(0) \neq 0$.

If the Δ-function $F(t)$ has zeros in $|t| \leqq q$, they will occur in nests of k, so their number will be a multiple of k, say rk. We define r to be the *order of* $F(t)$ (in $|t| \leqq q$).

LEMMA 4.8. *If the Δ-function $F(t)$ is of order zero there exists an admissible system $\mathcal{C}$ (necessarily of order zero) such that*

$$\Delta_C(t) = F(t).$$

For, we need only choose

$$C_0(t) = \frac{1}{\Omega_k} F(t); \qquad C_j(t) = 1, \qquad j = 1, \cdots, k-1,$$

and apply (2.11) and (2.33).

LEMMA 4.9. *If the Δ-function $F(t)$ is nonsingular and of order one, there is an admissible system $\mathcal{B}$ for which*

$$\Delta_B(t) = F(t).$$

We have $F(t) = (t^k - \alpha^k)P(t)$, where $P(t)$ is a Δ-function of order zero and $\alpha \neq 0$. From the proof of Theorem 3.3 (see (3.24)) we know there is an admissible $\mathcal{B}_1$ of order one for which $\Delta_{B_1}(t) = t^k - \alpha^k$. Also by Lemma 4.8 there is a $\mathcal{C}$, of order zero, with $\Delta_C(t)=(1/\Omega_k')P(t)$. On applying (3.11) we see that if $\mathcal{B}=\mathcal{B}_1\mathcal{C}$, then $\Delta_B(t)=F(t)$.

THEOREM 4.1. *Let the Δ-function $F(t)$ be nonsingular. There is an admissible system $\mathcal{A}$ for which*

$$\Delta_A(t) = F(t). \tag{4.11}$$

The order r of $F(t)$ may be assumed to exceed one. We can express F in the form

$$F(t) = P(t)\cdot\prod_{j=1}^{r}(t^k-\alpha_j^k) \qquad (\alpha_j\neq 0)$$

where $P(t)$ is a Δ-function of zero order. Now first order systems $\mathcal{B}_j$ $(j=1,\cdots,r)$ exist such that $\Delta_{B_j}(t)=t^k-\alpha_j^k$, and zero order system $\mathcal{C}$ such that $\Delta_C(t)=(1/\Omega_k')P(t)$. On setting $\mathcal{A}=\mathcal{B}_1\cdots\mathcal{B}_r\mathcal{C}$, we find from (3.19) that (4.11) holds.

If the condition of nonsingularity is lifted, a corresponding result holds:

THEOREM 4.2. *If F is a Δ-function, there is a system $\mathcal{A}$ in Γ satisfying* (4.11).

We need only consider a singular F of (positive) order r:

$$F(t) = t^{mk}P(t)\prod_{j=1}^{r-m}(t^k-\alpha_j^k) \qquad (\alpha_j\neq 0).$$

Let $\mathcal{D}$ be defined by

$$\mathcal{D}:\quad D_n(t) = ct^m \qquad (n=0,1,\cdots;\ c=(-1)^{(1-k)m/k}).$$

Then

$$\Delta_D(t) = \frac{t^{mk}}{\Omega_k'}\cdot$$

Hence if $\mathcal{A}_1=\mathcal{B}_1\cdots\mathcal{B}_{r-m}\mathcal{C}$ is chosen (as in Theorem 4.1) so that

$$\Delta_{A_1}(t) = P(t)\prod_{j=1}^{r-m}(t^k-\alpha_j^k),$$

then $\mathcal{A}=\mathcal{A}_1\mathcal{D}$ has the property (4.11).

LEMMA 4.10. *Let a be a nonzero constant. There are infinitely many admissible systems $\mathcal{C}$ (necessarily of order zero) for which*

$$\Delta_C(t) = a.$$

Define $\mathcal{C}$ by

$$C_j(t) = \lambda_j \qquad (j = 0, 1, \cdots, k-1),$$

where the λ's are as yet arbitrary but are to be nonzero. Then

$$\Delta_C(t) = \Omega_k \cdot \prod_{j=0}^{k-1} \lambda_j.$$

Choosing $\lambda_1, \cdots, \lambda_{k-1}$ arbitrarily (but different from zero), λ_0 may then be determined so that $\Delta_C(t) = a$. The lemma now follows from the fact that two non-identical sets of λ's give rise to two different systems $\mathcal{C}$.

THEOREM 4.3. *If F is a Δ-function, there exist infinitely many systems $\mathcal{A}$ in Γ (admissible if F is nonsingular) such that* (4.11) *holds.*

Let $\mathcal{A}_1$ be one such system (guaranteed by Theorem 4.2), and let $\mathcal{C}$ (which will be of order zero) be such that $\Delta_C(t) = 1/\Omega_k'$. Then $\mathcal{A} = \mathcal{C}\mathcal{A}_1$ also satisfies (4.11). Now two *different* $\mathcal{C}$'s will give rise to two different $\mathcal{A}$'s. For if F is nonsingular, then $\mathcal{A}_1$ is admissible, so if $\mathcal{C}_1\mathcal{A}_1 = \mathcal{C}_2\mathcal{A}_1$ then by the uniqueness of Lemma 4.4 we conclude that $\mathcal{C}_1 = \mathcal{C}_2$.

Now suppose F is singular. From the proof of Theorem 4.2, $\mathcal{A}_1$ has the form $\mathcal{A}_1 = \mathcal{A}_2\mathcal{D}$ where $\mathcal{A}_2$ is admissible. Hence $\mathcal{C}_1\mathcal{A}_1 = \mathcal{C}_2\mathcal{A}_1$ implies $\mathcal{C}_1\mathcal{A}_2\mathcal{D} = \mathcal{C}_2\mathcal{A}_2\mathcal{D}$. Let $\mathcal{B}_i = \mathcal{C}_i\mathcal{A}_2$ $(i = 1, 2)$. Then

$$\text{(a)} \qquad \sum_{j=0}^{\infty} b_{nj}^{(1)} t^j D_{n+j}(t) = \sum_{j=0}^{\infty} b_{nj}^{(2)} t^j D_{n+j}(t).$$

But from the definition of $\mathcal{D}$, $D_n(t) = ct^m$. On cancelling the common factor ct^m in (a) we obtain $\sum_0^\infty b_{nj}^{(1)} t^j = \sum_0^\infty b_{nj}^{(2)} t^j$; that is, $B_n^{(1)}(t) = B_n^{(2)}(t)$, so $\mathcal{B}_1 = \mathcal{B}_2$. This makes $\mathcal{C}_1\mathcal{A}_2 = \mathcal{C}_2\mathcal{A}_2$. The first part of the proof now applies, giving $\mathcal{C}_1 = \mathcal{C}_2$.

Hence whether F is singular or not, the condition that $\mathcal{C}_1 \neq \mathcal{C}_2$ is contradicted by the supposition that $\mathcal{C}_1\mathcal{A}_1 = \mathcal{C}_2\mathcal{A}_1$. This establishes the theorem.

In Theorem 3.5 we obtained a factorization of the admissible system $\mathcal{A}$. We come now to a useful modification of (3.41).

DEFINITION. A first order admissible system $\mathcal{B}$ is *semi-canonical* if an integer s $(0 \leqq s \leqq k-1)$, a number α $(0 < |\alpha| \leqq q)$, and numbers $\mu_0, \cdots, \mu_k$ exist such that

$$(4.12) \qquad \mathcal{B}: \quad B_j(t) = \begin{cases} 1, & j \neq s \ (j = 0, 1, \cdots, k-1); \\ \sum_{r=0}^{k} \mu_r t^r, & j = s, \end{cases}$$

where

$$\mu_0 \mu_k \neq 0, \qquad \mu_0/\mu_k = -\alpha^k;$$

and $\mathcal{B}$ is *canonical* if

$$(4.13) \qquad \mathcal{B}: \quad B_j(t) = \begin{cases} 1, & j \neq s \ (j = 0, 1, \cdots, k-1); \\ t^k - \alpha^k, & j = s. \end{cases}$$

From the proof of Theorem 3.1 it is seen that the systems $\mathcal{B}_1, \cdots, \mathcal{B}_l$ of Theorem 3.5 are semi-canonical.

LEMMA 4.11. *If $\mathcal{B}_1$, $\mathcal{B}_2$ are semi-canonical, and correspond to the same index s and zero α, then an admissible system $\mathcal{C}$ of order zero exists such that*

$$\mathcal{C}\mathcal{B}_1 = \mathcal{B}_2.$$

To show this, let $\mathcal{B}_i$: $\{B_{in}(t)\}$ $(i=1, 2)$ be given by

$$B_{i,s}(t) = \sum_{j=0}^{k} \mu_{i,k}t^k \qquad \left[\frac{\mu_{10}}{\mu_{1k}} = \frac{\mu_{20}}{\mu_{2k}} = -\alpha^k\right],$$

$$B_{i,r}(t) = 1 \qquad (r \neq s, r = 0, 1, \cdots, k-1).$$

We are to have

$$B_{2,j}(t) = c_{j0}B_{1,j}(t) + c_{j1}tB_{1,j+1}(t) + \cdots \qquad (j = 0, 1, \cdots, k-1).$$

For $j \neq s$ this becomes

$$1 = \sum_{r=0}^{\infty} c_{jr}t^r B_{1,j+r}(t),$$

so

$$c_{j0} = 1, \qquad c_{jr} = 0 \qquad (r > 0);$$

that is,

$$C_j(t) = 1 \qquad (j \neq s; j = 0, 1, \cdots, k-1).$$

And for $j = s$:

$$\sum_{r=0}^{k} \mu_{2,r}t^r = c_{s0}\left(\sum_{0}^{k} \mu_{1,r}t^r\right) + \sum_{p=1}^{k-1} c_{sp}t^p$$
$$+ c_{sk}t^k\left(\sum_{0}^{k} \mu_{1,r}t^r\right) + \sum_{p=1}^{k-1} c_{s,k+p}t^{k+p} + \cdots .$$

From this relation we obtain the equations

$$\mu_{20} = c_{s0}\mu_{1,0}; \qquad \mu_{2,p} = c_{s0}\mu_{1,p} + c_{s,p} \qquad (p = 1, \cdots, k-1);$$
$$\mu_{2,k} = c_{s0}\mu_{1,k} + c_{sk}\mu_{10}; \qquad 0 = c_{sk}\mu_{11} + c_{s,k+1};$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

And from these equations we find that

$$c_{s0} = \mu_{20}/\mu_{10} \neq 0; \qquad c_{s,p} = \mu_{2,p} - c_{s0}\mu_{1,p} \qquad (p = 1, \cdots, k-1);$$
$$c_{s,k+n} = 0 \qquad (n = 0, 1, \cdots).$$

Thus $C_s(t)$ is a polynomial (of degree not exceeding $k-1$):

$$C_s(t) = \frac{\mu_{20}}{\mu_{10}} + \frac{1}{\mu_{10}} \sum_{r=1}^{k-1} (\mu_{2,r}\mu_{10} - \mu_{20}\mu_{1,r})t^r.$$

Consequently, $\mathcal{C}$ is admissible and $\mathcal{C}B_1 = B_2$.

Now each of $\Delta_{B_1}(t)$ and $\Delta_{B_2}(t)$ has exactly k zeros. It therefore follows from Theorem 3.1 that $\Delta_C(t)$ has no zeros, so that $\mathcal{C}$ is of order zero. Thus the lemma is established.

Choosing $\mu_{20} = -\alpha^k$, $\mu_{2,k} = 1$, we obtain the following lemma.

LEMMA 4.12. *If $\mathcal{B}_1$ is semi-canonical, corresponding to the index s and the zero α, there is a unique admissible system $\mathcal{C}$ (necessarily of order zero) such that $\mathcal{C}\mathcal{B}_1$ is the canonical system corresponding to the same s and α.*

We know that $\mathcal{C}$ exists. That it is unique is a consequence of Lemma 4.4.

LEMMA 4.13. *If $\mathcal{A}$ is admissible of order one, then*

$$\mathcal{A} = \mathcal{C}_1\mathcal{B}\mathcal{C}_2 \tag{4.14}$$

where $\mathcal{C}_1$, $\mathcal{C}_2$ are admissible of order zero and $\mathcal{B}$ is canonical.

For by Theorem 3.3, $\mathcal{A} = \mathcal{B}_1\mathcal{C}_2$ where $\mathcal{B}_1$ is semi-canonical and $\mathcal{C}_2$ is of order zero. Also, there is a $\mathcal{C}_3$ such that $\mathcal{C}_3\mathcal{B}_1 = \mathcal{B}$, $\mathcal{B}$ canonical and $\mathcal{C}_3$ of order zero. Applying $\mathcal{C}_3^{-1}$ to both sides: $\mathcal{B}_1 = \mathcal{C}_1\mathcal{B}$, with $\mathcal{C}_1 = \mathcal{C}_3^{-1}$, so (4.14) holds.

LEMMA 4.14. *Let $\mathcal{B}_1$, $\mathcal{B}_2$ be canonical systems corresponding to the same zero α and to the respective integers 0, r. There exist zero order admissible systems $\mathcal{C}_i$ $(i=1, 2, 3, 4)$ such that*

$$\mathcal{C}_1\mathcal{B}_1\mathcal{C}_2 = \mathcal{B}_2, \qquad \mathcal{B}_1 = \mathcal{C}_3\mathcal{B}_2\mathcal{C}_4. \tag{4.15}$$

If $r=0$ we need only take $\mathcal{C}_i = \mathfrak{J}$ $(i=1, \cdots, 4)$. We may therefore suppose that $0<r<k$. Each of (4.15) clearly implies the other, so we need only consider the first relation, which can be written in the equivalent form

$$\mathcal{C}_1\mathcal{B}_1 = \mathcal{B}_2\mathcal{C}_3 \qquad [\mathcal{C}_3 = \mathcal{C}_2^{-1}]. \tag{4.16}$$

Then, apart from the questions of admissibility and order, we are to determine $\mathcal{C}_1$, $\mathcal{C}_3$ so that

$$\text{(a)}\quad \begin{aligned} &\overset{(1)}{c_{j0}} B_{1,j}(t) + \overset{(1)}{c_{j1}} tB_{1,j+1}(t) + \cdots = C_{3,j}(t) \quad [j \neq r; j = 0, 1, \cdots, k-1]; \\ &\overset{(1)}{c_{r0}} B_{1,r}(t) + \overset{(1)}{c_{r1}} tB_{1,r+1}(t) + \cdots \\ &\qquad\qquad = -\alpha^k C_{3,r}(t) + t^k C_{3,r+k}(t) = (t^k - \alpha^k)C_{3,r}(t). \end{aligned}$$

Now

$$B_{1,nk}(t) = t^k - \alpha^k; \qquad B_{1,nk+p}(t) = 1 \qquad\qquad (0 < p < k).$$

Hence if we write

$$C_{1,r}(t) = \sum_{j \not\equiv -r} c_{rj}^{(1)} t^j + \sum_{j \equiv -r} c_{rj}^{(1)} t^j = U(t) + V(t),$$

where the congruence signs refer to mod k, then

$$(t^k - \alpha^k) C_{3,r}(t) = U(t) + (t^k - \alpha^k) V(t).$$

It is seen that we want $U(t)$ to have the factor $(t^k - \alpha^k)$. For this purpose we take $C_{1,r}(t)$ to have the form

$$C_{1,r}(t) = (t^k - \alpha^k) U_0(t) + V(t) = U(t) + V(t),$$

where the power series for $U_0(t)$ contains no power of t that is congruent to $-r$ (mod k). In fact, *choose* $U_0(t) = 1$ and $V(t) = t^{k-r}$, so that

$$C_{1,r}(t) = -\alpha^k + t^{k-r} + t^k. \tag{4.17}$$

As for $C_{1,j}(t)$ $[j \neq r]$, define it as follows:

$$C_{1,j}(t) = 1 \quad (j \neq 0, r); \qquad C_{1,0}(t) = 1 + t^r. \tag{4.18}$$

This makes $\mathcal{C}_1$ admissible. System $\mathcal{C}_3$ is now completely determined by equations (a):

$$\begin{aligned} C_{3,j}(t) &= B_{1j}(t) = 1 \qquad\qquad\qquad\qquad (j \neq 0, r; j = 1, \cdots, k-1); \\ C_{3,0}(t) &= B_{10}(t) + t^r = -\alpha^k + t^r + t^k; \\ C_{3,r}(t) &= \frac{1}{(t^k - \alpha^k)} \left\{ -\alpha^k B_{1,r}(t) + t^{k-r} B_{1,k}(t) + t^k B_{1,r+k}(t) \right\} = 1 + t^{k-r}; \end{aligned} \tag{4.19}$$

and is of course admissible.

There remains to consider order. If $\mathcal{C}_1$ is of order zero, (4.16) shows that $\mathcal{C}_3$ has the same property. We need therefore only consider $\mathcal{C}_1$. For it we have

$$\Delta_{C_1}(t) = \begin{vmatrix} 1+t^r & 1 \cdots 1 & [(t^k-\alpha^k)+t^{k-r}] & 1 \cdots 1 \\ 1+\omega^r t^r & \omega \cdots \omega^{r-1} & \omega^r[(t^k-\alpha^k)+\omega^{-r}t^{k-r}] & \omega^{r+1} \cdots \omega^{k-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1+\omega^{r(k-1)}t^r & \omega^{k-1} \cdots \omega^{(r-1)(k-1)} & \omega^{r(k-1)}[(t^k-\alpha^k)+\omega^{-r(k-1)}t^{k-r}] & \omega^{(r+1)(k-1)} \cdots \omega^{(k-1)(k-1)} \end{vmatrix}.$$

This can be written as the sum of four determinants by splitting the 0th and rth columns in an obvious way; and it is readily seen that two of the new determinants are zero, so that

$$\Delta_{C_1}(t) = (t^k - \alpha^k)\Omega_k - t^r \cdot t^{k-r}\Omega_k = -\alpha^k \Omega_k.$$

As this is a (nonzero) constant, $\mathcal{C}_1$ (and with it $\mathcal{C}_3$) is of order zero. The lemma is therefore proved.

We now extend the scope of Lemma 4.14:

LEMMA 4.15. *If* $\mathcal{B}_1$, $\mathcal{B}_2$ *are canonical systems corresponding to the same zero* α *and to the respective integers* s, r, *then zero order admissible systems* $\mathcal{C}_i$ $(i=1, \cdots, 4)$ *exist so that*

$$\mathcal{C}_1\mathcal{B}_1\mathcal{C}_2 = \mathcal{B}_2, \qquad \mathcal{B}_1 = \mathcal{C}_3\mathcal{B}_2\mathcal{C}_4. \tag{4.20}$$

We need only establish the first of these relations. Let $\mathcal{B}^*$ be the canonical system corresponding to α and to the integer 0. Then we have

$$\mathcal{B}_2 = \mathcal{C}_5\mathcal{B}^*\mathcal{C}_6, \qquad \mathcal{B}^* = \mathcal{C}_7\mathcal{B}_1\mathcal{C}_8$$

where all $\mathcal{C}$'s are of order zero. Hence (4.20) holds with

$$\mathcal{C}_1 \equiv \mathcal{C}_5\mathcal{C}_7, \qquad \mathcal{C}_2 \equiv \mathcal{C}_8\mathcal{C}_6.$$

This lemma permits us to strengthen Lemma 4.13 as follows:

LEMMA 4.16. *Let* $\mathcal{A}$ *be admissible of order one, and let the integer* s $(0 \leqq s \leqq k-1)$ *be prescribed. Then*

$$\mathcal{A} = \mathcal{C}_1\mathcal{B}\mathcal{C}_2 \tag{4.21}$$

where $\mathcal{B}$ *is canonical and corresponds to the integer* s, *and* $\mathcal{C}_1$, $\mathcal{C}_2$ *are admissible of order zero.*

From this set of lemmas we are able to prove the following result.

THEOREM 4.4. *Let* $\mathcal{A}$ *be admissible of positive order* l, *and let its zeros be* $\{\omega^f\alpha_j\}$ $(f=0, 1, \cdots, k-1; j=1, \cdots, l)$. *Let the integers* s_j $(j=1, \cdots, l)$, *not necessarily distinct, be prescribed in* $0 \leqq s_j \leqq k-1$. *Then* $\mathcal{A}$ *has the factorization*

$$\mathcal{A} = \mathcal{C}_1\mathcal{B}_1\mathcal{C}_2\mathcal{B}_2 \cdots \mathcal{C}_l\mathcal{B}_l\mathcal{C}_{l+1}, \tag{4.22}$$

where all $\mathcal{C}$'s *are admissible of order zero, and where* $\mathcal{B}_j$ *is canonical and corresponds to the integer* s_j *and the zero* α_j.

To see this, we recall that Theorem 3.5 gives us the product

$$\mathcal{A} = \mathcal{B}_1^*\mathcal{B}_2^* \cdots \mathcal{B}_l^*\mathcal{C}^*,$$

with $\mathcal{B}_j^*$ admissible of order one and $\mathcal{C}^*$ of order zero. Lemma 4.16 now applies to yield (4.22).

REMARK. The decomposition (4.22) for $\mathcal{A}$ is not unique, since we can permute the zeros α_j.

The remainder of this section concerns systems of equations having a solution that approaches zero. We shall need the results in §5.

LEMMA 4.17. *If an admissible system* $\mathcal{A}$ *is of order zero in* $|t| \leqq \rho$ *with* $\rho > 1$, *and if* $c_n \to 0$, *then the system of equations*

$$\mathcal{A}: A_n[X] = c_n \qquad (n = 0, 1, \cdots) \tag{4.23}$$

has a solution $X=\{x_n\}$ *with* $x_n \to 0$.

Since $((c_n)) \leqq 1 < \rho$, the operator $\mathcal{A}^{-1}$ can be applied to (4.23). It gives the *equivalent* system

$$x_n = \alpha_{n0}c_n + \alpha_{n1}c_{n+1} + \cdots \qquad (n = 0, 1, \cdots),$$

where $\{\alpha_{nj}\}$ is the matrix for $\mathcal{A}^{-1}$. Now $\mathcal{A}^{-1}$ is admissible of order zero in $|t| \leqq \rho$, so constants M, θ exist, with $\theta < 1/\rho < 1$, such that

$$|\alpha_{nj}| \leqq M\theta^j.$$

Define $\{h_n\}$ by

$$h_n = \max \{ |c_n|, |c_{n+1}|, \cdots \}.$$

Then

$$|x_n| \leqq \frac{Mh_n}{1-\theta},$$

and since $h_n \to 0$, so does x_n.

LEMMA 4.18. *Let* $\mathcal{A}$ *be of order one and canonical in* $|t| \leqq \rho$ *with* $\rho > 1$, *and let the zeros of* $\Delta_A(t)$ *be* $\{\omega^f\alpha\}$ $(f=0, 1, \cdots, k-1)$. *If* $|\alpha| \neq 1$, *and* $c_n \to 0$, *then* (4.23) *has a solution* X *with* $x_n \to 0$.

If $|\alpha| > 1$, choose ρ' in $1 < \rho' < |\alpha|$. $\mathcal{A}$ will be of order zero in $|t| \leqq \rho'$, so we can apply Lemma 4.17. We may therefore suppose that $|\alpha| < 1$. By hypothesis there is an s for which

$$A_j(t) = \begin{cases} t^k - \alpha^k, & j \equiv s \\ 1, & j \not\equiv s \end{cases} \pmod{k};$$

so (4.23) becomes

$$(4.24) \qquad \begin{aligned} x_r &= c_r & (r \not\equiv s); \\ -\alpha^k x_{s+nk} + x_{s+(n+1)k} &= c_{s+nk} & (n = 0, 1, \cdots). \end{aligned}$$

We see that $x_r \to 0$ so long as r ranges over values non-congruent to s. As for x_{s+nk}, set

$$(4.25) \qquad y_n = x_{s+nk}, \qquad \delta_n = c_{s+nk} \qquad (\delta_n \to 0).$$

Then (compare (3.36)–(3.40))

$$(4.26) \qquad -\alpha^k y_n + y_{n+1} = \delta_n,$$

and

$$(4.27) \qquad y_n = \alpha^{nk} y_0 + \sum_{j=0}^{n-1} \alpha^{jk} \delta_{n-j-1}.$$

Now $|\alpha| < 1$, so $\alpha^{nk}y_0 \to 0$. Let

$$d_n = \max \{ |\delta_n|, |\delta_{n+1}|, \cdots \} \qquad (d_n \to 0),$$

and choose λ so that $|\delta_n| < \lambda$.

There exists an integer p such that

$$\sum_{j=p}^{n-1} |\alpha^{jk}| \leqq \frac{|\alpha|^{pk}}{1 - |\alpha|^k} < \frac{\epsilon}{2\lambda},$$

so

$$\sum_{j=p}^{n-1} |\alpha^{jk}\delta_{n-j-1}| < \frac{\epsilon}{2}.$$

Also,

$$\sum_{j=0}^{p-1} |\alpha^{jk}\delta_{n-j-1}| \leqq d_{n-p} \sum_{0}^{p-1} |\alpha|^{jk} < \frac{d_{n-p}}{1 - |\alpha|^k} < \frac{\epsilon}{2}$$

for all $n > N$ (N suitably chosen). Hence $y_n \to 0$. It follows that $x_n \to 0$.

REMARK. If $|\alpha| = 1$ the conclusion of the above lemma (and of the theorem that follows) need not hold. This is seen from the following examples:

(i) k even. Take $\alpha = -1$, $\delta_n = 1/(n+1)$. Then $y_n = y_0 + \sum_{j=1}^{n} 1/j$.

(ii) k odd. Take $\alpha = -1$, $\delta_n = (-1)^{n+1}/(n+1)$. Then $y_n = (-1)^n[y_0 + \sum_{j=1}^{n} 1/j]$.

In neither case does $y_n \to 0$. This possible failure when $|\alpha| = 1$ has an important bearing (an unfavorable one, be it noted) on the factorization considered in §5.

THEOREM 4.5. *Let $\mathcal{A}$ be admissible in $|t| \leqq \rho$ with $\rho > 1$, and suppose no zero of $\Delta_A(t)$ is of magnitude one. If $c_n \to 0$ there is a solution X of* (4.23) *for which $x_n \to 0$.*

In view of Lemma 4.17 we may suppose that the order l of $\mathcal{A}$ is positive. Applying Theorem 4.4 we can write

$$\mathcal{A} = \mathcal{C}_1\mathcal{B}_1\mathcal{C}_2 \cdots \mathcal{C}_l\mathcal{B}_l\mathcal{C}_{l+1}$$

where the $\mathcal{B}$'s and $\mathcal{C}$'s have the properties of that theorem. Let $\mathcal{B}_i$, $\mathcal{C}_i$ be given by

$$\mathcal{B}_i\colon \ \{B_{i,n}(t)\}; \qquad \mathcal{C}_i\colon \ \{C_{i,n}(t)\}.$$

By the two preceding lemmas,

$$\mathcal{C}_1\colon \ C_{1,n}[U^{(1)}] = c_n$$

has a solution $U^{(1)}\colon \{u_n^{(1)}\}$ with $u_n^{(1)} \to 0$;

$$\mathcal{B}_1: \quad B_{1,n}[U^{(2)}] = u_n^{(1)}$$

has a solution $U^{(2)}: \{u_n^{(2)}\}$ with $u_n^{(2)} \to 0$; and so on, until finally

$$\mathcal{C}_{l+1}: \quad C_{l+1,n}[U^{(2l+1)}] = u_n^{(2l)}$$

has a solution $U^{(2l+1)}: \{u_n^{(2l+1)}\}$ with $u_n^{(2l+1)} \to 0$. Now $X = U^{(2l+1)}$ satisfies (4.23), so the theorem is established.

5. **The perturbation case: Factorization.** We recall that a system $\mathcal{A}$ is *k-periodic* if

$$A_{nk+j}(t) = A_j(t), \qquad j = 0, 1, \cdots, k-1;$$

and is *admissible* if it is k-periodic and $A_n(t)$ is analytic in $|t| \leqq q$, with $A_n(0) \neq 0$ for all n. We now define an *admissible system of perturbation terms*

$$\mathcal{A}^*: \quad A_n^*[X] \equiv \sum_{j=0}^{\infty} a_{n,j}^* x_{n+j} \qquad (n = 0, 1, \cdots) \tag{5.1}$$

to be a system with the following property:

$$| a_{n,j}^* | \leqq k_n \theta^j \tag{5.2}$$

with

$$k_n \to 0, \tag{5.3}$$

$$\theta < 1/q. \tag{5.4}$$

Note that this makes the functions $A_n^*(t)$ analytic in $|t| \leqq q$. For certain computations that we must make, it is desirable that sequence $\{k_n\}$ shall approach zero monotonically. Now given $\{k_n\}$ with $0 < k_n$ and $k_n \to 0$, there exists a sequence $\{k_n'\}$ such that $k_n \leqq k_n'$, $k_n' \downarrow 0$; and k_n' may be used in (5.2) in place of k_n. *It is therefore no restriction to suppose, as we do from now on, that* $k_n \downarrow 0$.

We reserve the asterisk superscript for systems of perturbation terms, so when a system is said to be *admissible*, it will be clear from the notation whether the system is k-periodic, or is a system of perturbation terms, or is an *admissible perturbation system*, which we define as follows:

Definition. A system is an *admissible perturbation system* if it has the form $\mathcal{A} + \mathcal{A}^*$, where $\mathcal{A}$ and $\mathcal{A}^*$ are both admissible, *and where*

$$a_{n0} + a_{n0}^* \neq 0 \qquad (n = 0, 1, \cdots). \tag{5.5}$$

A perturbation system of equations is of form (1.6).

Lemma 5.1. *If $\mathcal{C}$ and $\mathcal{A}^*$ are admissible, then $\mathcal{C}\mathcal{A}^*$ and $\mathcal{A}^*\mathcal{C}$ are both admissible systems of perturbation terms.*

Let $\mathcal{D}^* = \mathcal{A}^*\mathcal{C}$, so that

$$D_n^*(t) = \overset{*}{a}_{n0}C_n(t) + \overset{*}{a}_{n1}tC_{n+1}(t) + \cdots .$$

By hypothesis $\{a_{nj}^*\}$ satisfies (5.2) for certain θ, $\{k_n\}$ for which (5.3), (5.4) hold; and $|c_{nj}| \leqq M\theta'^j$ for suitable M, θ' with $\theta' < 1/q$. Hence

$$|d_{nj}^*| \leqq \sum_{r=0}^{j} |\overset{*}{a}_{nr}c_{n+r,j-r}| \leqq Mk_n(j+1)\theta''^j,$$

where $\theta'' = \max\{\theta, \theta'\} < 1/q$. Choose θ''' in the range $\theta'' < \theta''' < 1/q$. Then J exists such that for all $j > J$, $(j+1)\theta''^j < \theta'''^j$; so M' can be found for which

$$|d_{nj}^*| \leqq (M'k_n)\theta'''^j$$

for all n and j. Accordingly, $\mathcal{D}^*$ is an admissible system of perturbation terms.

Similarly, if $\mathcal{E}^* = \mathcal{C}\mathcal{A}^*$, then

$$|e_{nj}^*| \leqq Mk_n(j+1)\theta''^j \qquad (\theta'' = \max\{\theta, \theta'\});$$

so again the desired conclusion follows.

Let $\mathcal{A} + \mathcal{A}^*$ be admissible. We consider the problem of its factorization. If $\mathcal{A}$ is of order zero, then $\mathcal{A} + \mathcal{A}^*$ is essentially in factored form as it stands, since we can write

$$\mathcal{A} + \mathcal{A}^* = \mathcal{A}(\mathfrak{I} + \mathcal{A}^{-1}\mathcal{A}^*) = \mathcal{A}(\mathfrak{I} + \mathcal{B}^*)$$

where $\mathfrak{I}$ is the identity and $\mathcal{B}^* = \mathcal{A}^{-1}\mathcal{A}^*$ is admissible. We need therefore only consider the case that $\mathcal{A}$ is of positive order l.

The function $\Delta_A(t)$ has then l nests of zeros: $\{\omega^f\alpha_j\}$, $f = 0, 1, \cdots, k-1$; $j = 1, \cdots, l$. We may order the zeros so that

$$|\alpha_1| \geqq |\alpha_2| \geqq \cdots \geqq |\alpha_l|. \tag{5.6}$$

We make the special assumption that

$$|\alpha_l| < |\alpha_j|, \qquad j < l; \tag{5.7}$$

and for simplicity write α for α_l.

By (4.22) of Theorem 4.4 we have

$$\mathcal{A} = \mathcal{C}_1\mathcal{B}_1\mathcal{C}_2\mathcal{B}_2 \cdots \mathcal{C}_l\mathcal{B}_l\mathcal{C}_{l+1}, \tag{5.8}$$

where all systems are admissible, the $\mathcal{C}$'s of order zero and the $\mathcal{B}$'s canonical. Now

$$\mathcal{A} + \mathcal{A}^* = (\mathcal{C}_1 \cdots \mathcal{B}_l + \mathcal{A}^*\mathcal{C}_{l+1}^{-1})\mathcal{C}_{l+1} \equiv (\mathcal{A}_1 + \mathcal{E}^*)\mathcal{C}_{l+1}, \tag{5.9}$$

where

$$\mathcal{A}_1 = \mathcal{C}_1 \cdots \mathcal{B}_l = \mathcal{B}\mathcal{B}_l = \mathcal{B}\mathcal{C}, \qquad \mathcal{E}^* = \mathcal{A}^*\mathcal{C}_{l+1}^{-1}$$

and

$$\mathcal{C} = \mathcal{B}_l, \qquad \mathcal{B} = \mathcal{C}_1\mathcal{B}_1 \cdots \mathcal{C}_l.$$

System $\mathcal{E}^*$ is admissible.

$\mathcal{A}+\mathcal{A}^*$ will be factorable if $\mathcal{A}_1+\mathcal{E}^*$ is, so we consider this last system. We introduce a parameter λ, and investigate the possibility of a factorization of form

$$\mathcal{A}_1 + \lambda\mathcal{E}^* = \mathcal{B}\mathcal{C} + \lambda\mathcal{E}^* = \left(\mathcal{B} + \sum_{r=1}^{\infty} \lambda^r \mathcal{B}_r^*\right)\left(\mathcal{C} + \sum_{r=1}^{\infty} \lambda^r \mathcal{C}_r^*\right). \tag{5.10}$$

On expanding and equating like powers of λ, we have

$$\begin{aligned} \mathcal{E}^* &= \mathcal{B}\mathcal{C}_1^* + \mathcal{B}_1^*\mathcal{C}, \\ 0 &= \mathcal{B}\mathcal{C}_r^* + \mathcal{B}_1^*\mathcal{C}_{r-1}^* + \mathcal{B}_2^*\mathcal{C}_{r-2}^* + \cdots + \mathcal{B}_r^*\mathcal{C} \qquad (r = 2, 3, \cdots); \end{aligned} \tag{5.11}$$

and on defining $\mathcal{D}_r^*$ $(r=1, 2, \cdots)$ by

$$\begin{aligned} \mathcal{D}_1^* &= \mathcal{E}^*, \\ \mathcal{D}_r^* &= -\left\{\mathcal{B}_1^*\mathcal{C}_{r-1}^* + \cdots + \mathcal{B}_{r-1}^*\mathcal{C}_1^*\right\} \qquad (r = 2, 3, \cdots), \end{aligned} \tag{5.12}$$

then

$$\mathcal{D}_r^* = \mathcal{B}\mathcal{C}_r^* + \mathcal{B}_r^*\mathcal{C} \qquad (r = 1, 2, \cdots). \tag{5.13}$$

Recall that $\mathcal{C}=\mathcal{B}_l$ is canonical, and to it corresponds the nest of zeros $\{\omega^j\alpha\}$, where $\alpha=\alpha_l$ satisfies (5.7). $\mathcal{C}$ is defined by

$$C_n(t) = \begin{cases} t^k - \alpha^k, & n \equiv s \\ 1, & n \not\equiv s \end{cases} \pmod{k}. \tag{5.14}$$

Write

$$\begin{aligned} &\mathcal{D}_r^*: \quad \{D_{r,n}^*(t)\}; \qquad \mathcal{B}_r^*: \quad \left\{B_{r,n}^*(t) = \sum_{j=0}^{\infty} \beta_{r,n;j}t^j\right\}; \\ &\mathcal{C}_r^*: \quad \{C_{r,n}^*(t)\} \qquad (r = 1, 2, \cdots; n = 0, 1, \cdots). \end{aligned} \tag{5.15}$$

For $r=1$ we obtain from (5.13) the relation

$$\begin{aligned} D_{1,n}^*(\omega^j t) &= \sum_{j=0}^{\infty} b_{n j}\omega^{j i}t^i C_{1,n+j}^*(\omega^j t) + \sum_{j=0}^{\infty} \beta_{1,n;j}\omega^{j i}t^i C_{n+j}(t), \\ &= 0, 1, \cdots, k-1. \end{aligned} \tag{5.16}$$

Define the functions $\Gamma_{1,n;j}(t)$ by

$$(5.17) \qquad \Gamma_{1,n;j}(t) = \sum_{r=0}^{\infty} \beta_{1,n;j+rk} t^{j+rk} \qquad (j = 0, 1, \cdots, k-1).$$

Then

$$(5.18) \qquad D_{1,n}^{*}(\omega^f t) = \sum_{j=0}^{\infty} b_{n j}\omega^{f j} t^j C_{1,n+j}^{*}(\omega^f t) + \sum_{j=0}^{k-1} \Gamma_{1,n;j}(t)\omega^{f j} C_{n+j}(t), \qquad (f = 0, 1, \cdots, k-1).$$

For n fixed, regard (5.18) as a system of linear equations in the Γ's. The determinant of this system is found from (5.14) to be $\Omega_k(t^k - \alpha^k)$, where Ω_k is the constant defined by (2.33). We therefore wish to choose the functions $C_{1,n+j}^{*}(t)$ so that, when we solve for the Γ's by Cramer's rule, the zeros of the denominator (namely $\omega^f\alpha$, $f = 0, 1, \cdots, k-1$) will cancel out. This will happen if we impose the conditions

$$(5.19) \qquad D_{1,n}^{*}(\omega^f t) - \sum_{j=0}^{\infty} b_{n j}\omega^{f j} t^j C_{1,n+j}^{*}(\omega^f t) = 0 \qquad (f = 0, 1, \cdots, k-1)$$

for $t = \omega^i\alpha$, $i = 0, 1, \cdots, k-1$.

Define $x_n^{(m)}$, $\delta_n^{(m)}$ by

$$(5.20) \qquad x_n^{(m)} = C_{1,n}^{*}(\omega^m \alpha),$$

$$(5.21) \qquad \delta_n^{(m)} = D_{1,n}^{*}(\omega^m \alpha).$$

Then (5.19) becomes

$$(5.22) \qquad \sum_{j=0}^{\infty} b_{n j}\omega^{m j} x_{n+j}^{(m)} \alpha^j = \delta_n^{(m)} \qquad (m = 0, 1, \cdots, k-1; n = 0, 1, \cdots).$$

For *m fixed*, (5.22) is a k-periodic system with determinant $\Delta_B(\omega^m \alpha t)$. Since $\Delta_B(t)$ has the zeros $\{\omega^f\alpha_j\}$, $j = 1, \cdots, l-1$, and $|\alpha_j| > |\alpha|$ by (5.7), the present Δ-function has all its zeros greater than one in magnitude.

Now $\mathcal{D}_1^* = \mathcal{E}^*$ is admissible. Hence

$$(5.23) \qquad D_{1,n}^{*}(t) \ll k_n \sum_{0}^{\infty} (\theta t)^i = \frac{k_n}{1 - \theta t} \qquad [|t| < 1/\theta, \theta < 1/q],$$

and

$$(5.24) \qquad |\delta_n^{(m)}| = |\overset{*}{D}_{1,n}(\omega^m \alpha)| \leqq \frac{k_n}{1 - \theta|\alpha|};$$

so

$$\delta_n^{(m)} \to 0.$$

By Theorem 4.5, therefore, (5.22) (m fixed) has a solution $X_m = \{x_n^{(m)}\}$ for which $x_n^{(m)} \to 0$ as $n \to \infty$.

There is a number $q' > 1$, independent of m, such that the system defined by the left side of (5.22) is of order zero in $|t| \leqq q'$. Consequently M and λ exist, *with* $\lambda < 1/q' < 1$, such that an upper bound for the magnitude of the (n, j)th coefficient of the matrix inverse to (5.22) is $M\lambda^j$, for all n, j and m. From the proof of Lemma 4.17 we conclude that

$$(5.25) \qquad | x_n^{(m)} | \leqq \frac{M}{1 - \lambda} h_n,$$

where $\{h_n\}$ is any sequence for which $h_n \to 0$ and such that

$$h_n \geqq \max \{ | \delta_{n+j}^{(m)} | \} \qquad (m = 0, 1, \cdots, k - 1; j = 0, 1 \cdots).$$

Now $k_n \downarrow 0$. We see therefore that we may choose

$$h_n = \frac{k_n}{1 - \theta |\alpha|},$$

and that $h_n \downarrow 0$. Then,

$$(5.26) \qquad | x_n^{(m)} | \leqq L_1 k_n \qquad \left[L_1 \equiv \frac{M}{(1 - \lambda)(1 - \theta |\alpha|)} \right];$$

so

$$(5.27) \qquad | C_{1,n}^{*}(\omega^m \alpha) | \leqq L_1 k_n \qquad (m = 0, 1, \cdots, k - 1; n = 0, 1, \cdots).$$

We have obtained values for $C_{1,n}^{*}(t)$ at the points $\omega^m \alpha$ $(m = 0, 1, \cdots, k-1)$. Now we complete the definition of these functions to make them as simple as possible: we take them as polynomials. In fact, set

$$(5.28) \quad C_{1,n}^{*}(t) = R_{1,n;0} + \sum_{p=0}^{k-2} (t - \alpha)(t - \omega\alpha) \cdots (t - \omega^p \alpha) R_{1,n;p+1},$$

where the R's are constants, uniquely determined recurrently by the condition that $C_{1,n}^{*}(t)$ takes on known values at $t = \omega^m \alpha$ $(m = 0, 1, \cdots, k-1)$.

From (5.27) and (5.28) it follows that constants $M_{1,0}, \cdots, M_{1,k-1}$ exist, *independent of* n, such that

$$| R_{1,n;j} | \leqq M_{1,j} k_n \qquad (j = 0, 1, \cdots, k - 1).$$

The M's can be expressed solely in terms of L_1 and $|\alpha|$. If we set

$$M = \max \{ M_{1,j} \},$$

then

$$| R_{1,n;j} | \leqq M_1 k_n.$$

We see from (5.28) that there is a constant σ, *whose value depends only on* α, such that

$$C^*_{1,n}(t) \ll \sigma M_1 k_n(1 + t + \cdots + t^{k-1}). \tag{5.29}$$

Let us return to (5.18). The functions $C^*_{1,n}(t)$ are analytic in $|t| \leqq q$, and from (5.29) we see that the first series in (5.18) converges uniformly in $|t| \leqq q'$ (q' some number exceeding q). Hence (5.18) defines (and uniquely) functions Γ that are analytic in $|t| \leqq q$. Before we can make the step from (5.18) back to (5.16), however, we must show that the functions Γ have power series of form (5.17); that is, that

$$\Gamma_{1,n;j}(\omega^f t) = \omega^{fj}\Gamma_{1,n;j}(t) \qquad (f, j = 0, 1, \cdots, k-1).$$

To establish this property, replace t by $\omega^p t$ in (5.18). We obtain

$$D^*_{1,n}(\omega^{f+p}t) = \sum_{j=0}^{\infty} b_{n,j}\omega^{(f+p)j}t^j C^*_{1,n+j}(\omega^{(f+p)}t) + \sum_{j=0}^{k-1}\left\{\omega^{-pj}\Gamma_{1,n;j}(\omega^p t)\right\}\omega^{(f+p)j}C_{n+j}(t).$$

This is the $(f+p)$th equation of (5.18), with the function $\Gamma_{1,n;j}(t)$ replaced by $\omega^{-pj}\Gamma_{1,n;j}(\omega^p t)$. Since (5.18) uniquely determines the Γ's, this requires that $\Gamma_{1,n;j}(t) = \omega^{-pj}\Gamma_{1,n;j}(\omega^p t)$, as was to be shown.

Consequently, we have shown the existence of two system $\mathcal{C}^*_1$, $\mathcal{B}^*_1$ for which $C^*_{1,n}(t)$, $B^*_{1,n}(t)$ are analytic in $|t| \leqq q$. We wish to prove that these systems are admissible. That $\mathcal{C}^*_1$ is so is shown by (5.29); the proof for $\mathcal{B}^*_1$ is more difficult.

LEMMA 5.2. *Let*

$$F(t) = \sum_{n=0}^{\infty} \gamma_n t^{n+l} + P_{l-1}(t), \tag{5.30}$$

with $P_{l-1}(t)$ a polynomial of degree not exceeding $l-1$, be analytic in $|t| \leqq p$, and suppose[11]

$$F(r_1) = \cdots = F(r_l) = 0; \qquad |r_j| \leqq p \qquad (j = 1, \cdots, l).$$

Define $Z(t)$ by

$$Z(t) = F(t)\left\{\prod_{j=1}^{l}(t - r_j)\right\}^{-1}.$$

Then

$$Z(t) = \sum_{n=0}^{\infty} \gamma_n H_n(t; r_1, \cdots, r_l), \tag{5.31}$$

(11) The r_α's need not be distinct provided that each set of equal r_α's is in number not greater than the multiplicity of their common value as a zero of $F(t)$.

where H_n is the symmetric homogeneous polynomial of degree n in $t; r_1, \cdots, r_l$, all of whose coefficients are unity; that is,

$$H_n(t; r_1, \cdots, r_l) = \sum r_1^{j_1} \cdots r_l^{j_l} t^j, \tag{5.32}$$

summed over all non-negative integers $j_1, \cdots, j_l, j$ for which $j_1 + \cdots + j_l + j = n$. Moreover, (5.31) *converges absolutely and uniformly for $|t| \leq p$.*

Since $F(r_1) = 0$,

$$Z_1(t) \equiv \frac{F(t)}{t - r_1} = \frac{1}{t - r_1}\left[\sum_{n=0}^{\infty} \gamma_n(t^{n+l} - r_1^{n+l}) + P_{l-1}(t) - P_{l-1}(r_1)\right]$$

$$= \sum_0^{\infty} \gamma_n \left\{\sum_{j=0}^{n+l-1} t^j r_1^{n+l-1-j}\right\} + P_{l-2}(t) = \sum_{n=0}^{\infty} \gamma_n H_{n+l-1}(t; r_1) + P_{l-2}(t),$$

where P_{l-2} is of degree not exceeding its index. Now $F(r_2) = 0$, so

$$Z_2(t) \equiv \frac{Z_1(t)}{t - r_2} = \frac{1}{t - r_2}\left[\sum_0^{\infty} \gamma_n \left\{\sum_{j=0}^{n+l-1} r_1^{n+l-1-j}(t^j - r_2^j)\right\} + P_{l-2}(t) - P_{l-2}(r_2)\right]$$

$$= \sum_0^{\infty} \gamma_n H_{n+l-2}(t; r_1, r_2) + P_{l-3}(t).$$

A simple induction permits this process to be continued until all the divisions have been carried out. The polynomial drops out completely, leaving the sum (5.31). Series $\sum \gamma_n t^{n+l}$ is absolutely and uniformly convergent in $|t| \leq p$. If each term in $H_{n+l-1}(t; r_1)$ is replaced by its absolute value we obtain a sum that does not exceed $(n+l)p^{n+l-1}$, so $\sum \gamma_n H_{n+l-1}(t; r_1)$ is absolutely and uniformly convergent in $|t| \leq p$. The same argument applies at each step, so the final series (5.31) has the stated convergence properties.

LEMMA 5.3. *Let*

$$J_{l,n} = \sum r_1^{j_1} \cdots r_l^{j_l} \equiv H_n(r_1, \cdots, r_l),$$

the summation being over all non-negative integers for which $j_1 + \cdots + j_l = n$. If

$$r = \max\{|r_1|, \cdots, |r_l|\},$$

then

$$|J_{l,n}| \leq r^n \binom{n + l - 1}{l - 1} = r^n \binom{n + l - 1}{n}.$$

This inequality is a consequence of the identity

$$\sum 1 = \binom{n + l - 1}{l - 1} \qquad (j_1 + \cdots + j_l = n),$$

which is readily proved, for example, by an induction.

We may now return to the functions Γ. Solving (5.18), we see from the character of the coefficient determinant that if $p \not\equiv s-n \pmod{k}$, then

$$(5.33) \qquad \Gamma_{1,n;p}(t) = \sum_{r=0}^{k-1} \lambda_{p,r}\left\{D^*_{1,n}(\omega^r t) - \sum_{j=0}^{\infty} b_{nj}\omega^{rj}t^j C^*_{1,n+j}(\omega^r t)\right\};$$

and if $i \equiv s-n \pmod{k}$, then

$$(5.34) \quad (t^k - \alpha^k)\Gamma_{1,n;i}(t) = \sum_{r=0}^{k-1} \lambda_{i,r}\left\{D^*_{1,n}(\omega^r t) - \sum_{j=0}^{\infty} b_{nj}\omega^{rj}t^j C^*_{1,n+j}(\omega^r t)\right\}.$$

In both cases the λ's are constants whose value is determined solely by k. Define Λ by

$$\Lambda = \max\left\{ |\lambda_{jr}| \right\} \qquad (j, r = 0, 1, \cdots, k-1).$$

Consider the first case. Since $\mathcal{B}$ is admissible we know that constants N, θ' exist such that

$$|b_{nj}| \leqq N\theta'^j \qquad (\theta' < 1/q).$$

Hence, using (5.23) and (5.29):

$$\Gamma_{1,n;p}(t) \ll k\Lambda\left\{k_n \sum_{j=0}^{\infty} (\theta t)^j + N\sigma M k_n \sum_{j=0}^{\infty} (\theta' t)^j (1 + t + \cdots + t^{k-1})\right\}.$$

Let

$$(5.35) \qquad \theta'' = \max\left\{\theta, \theta'\right\}, \qquad \zeta = \sum_{r=0}^{k-1} \theta''^{-r}.$$

Then

$$(5.36) \qquad \Gamma_{1,n;p}(t) \ll \frac{k\Lambda(1 + NM\sigma\zeta)}{1 - \theta'' t} k_n \equiv \frac{N_1 k_n}{1 - \theta'' t},$$

for $p \not\equiv s-n \pmod{k}$.

Now consider $i \equiv s-n$. The right side of (5.34) is clearly subject to the same inequality (5.36) as was found for the right side of (5.33). Hence if we write relation (5.34) in power series form:

$$(5.37) \qquad (t^k - \alpha^k)\Gamma_{1,n;i}(t) = \sum_{p=0}^{\infty} \mu_{i,n;p+k} t^{p+k} + P_{k-1}(t),$$

where P_{k-1} is a polynomial of degree not exceeding $k-1$, then

$$|\mu_{i,n;p}| \leqq N_1 \theta''^p k_n.$$

Since $\Gamma_{1,n;i}(t)$ is analytic in $|t| \leqq q$, the right side of (5.37) has the zeros

$t=\omega^r\alpha$, $r=0, 1, \cdots, k-1$. Consequently by Lemma 5.2,

$$\Gamma_{1,n;i}(t) = \sum_{p=0}^{\infty} \mu_{i,n;p+k} H_p(t; \alpha, \omega\alpha, \cdots, \omega^{k-1}\alpha),$$

where

$$H_p = \sum (\alpha)^{j_1} \cdots (\omega^{k-1}\alpha)^{j_k} t^j \qquad (j_1 + \cdots + j_k + j = p).$$

We can write

$$H_p = \sum_{j=0}^{p} t^j J_{k,p-j}$$

where

$$J_{k,p-j} = \sum (\alpha)^{j_1} \cdots (\omega^{k-1}\alpha)^{j_k} \qquad (j_1 + \cdots + j_k = p - j).$$

Now by Lemma 5.3,

$$| J_{k,p-j} | \leqq | \alpha |^{p-j} \binom{p-j+k-1}{p-j},$$

so

$$H_p \ll \sum_{j=0}^{p} \binom{p-j+k-1}{p-j} | \alpha |^{p-j} t^j.$$

Thus,

$$\Gamma_{1,n;i}(t) \ll N_1 \theta''^k k_n \sum_{j=0}^{\infty} \left(\frac{t}{|\alpha|}\right)^j \left\{ \sum_{p=j}^{\infty} \binom{p-j+k-1}{p-j} (\theta'' | \alpha |)^p \right\}$$

$$= N_1 \theta''^k k_n \sum_{j=0}^{\infty} (\theta'' t)^j \left\{ \sum_{r=0}^{\infty} \binom{r+k-1}{r} (\theta'' | \alpha |)^r \right\}.$$

Since $\theta''|\alpha| < 1$, the brace converges to the sum $(1-\theta''|\alpha|)^{-k}$. Therefore

$$\Gamma_{1,n;i}(t) \ll \frac{N_1\theta''^k}{(1-\theta''|\alpha|)^k} \cdot \frac{k_n}{1-\theta'' t} \equiv \frac{N_2 k_n}{1-\theta'' t} \qquad (i \equiv s - n \pmod{k}). \tag{5.38}$$

Combining (5.36) and (5.38), we have for *all* $j=0, 1, \cdots, k-1$,

$$\Gamma_{1,n;j}(t) \ll \frac{Qk_n}{1-\theta'' t} \tag{5.39}$$

where

$$Q = \max \{N_1, N_2\}.$$

From the definition (5.17) of $\Gamma_{1,n;j}$,

$$(5.40) \qquad B^*_{1,n}(t) = \sum_{j=0}^{\infty} \beta_{1,n;j}t^j = \sum_{r=0}^{k-1} \Gamma_{1,n;r}(t),$$

so

$$(5.41) \qquad B^*_{1,n}(t) \ll \frac{Qk_n}{1-\theta''t}.$$

This shows that $\mathcal{B}_1^*$ is admissible.

Now turn to (5.13) with $r=2$. $\mathcal{D}_2^* = -\mathcal{B}_1^*\mathcal{C}_1^*$, so from (5.41) and (5.29),

$$D^*_{2,n}(t) = -\sum_{j=0}^{\infty} \beta_{1,n;j}t^jC^*_{1,n+j}(t) \ll \sum_{j=0}^{\infty} (Qk_n\theta''^jt^j)\{\sigma M_1k_n(1+t+\cdots+t^{k-1})\}.$$

From this we obtain

$$(5.42) \qquad D^*_{2,n}(t) \ll \frac{\sigma MQ\zeta}{1-\theta''t}\cdot k_n^2 \equiv \frac{Lk_n^2}{1-\theta''t} \qquad (L \equiv \sigma MQ\zeta).$$

$\mathcal{D}_2^*$ is thus admissible. It is to be noted, in comparison with (5.23), which can be modified to read

$$D^*_{1,n}(t) \ll \frac{k_n}{1-\theta''t},$$

that the inequality for $D^*_{2,n}(t)$ has captured a *second factor* k_n.

Relation (5.13) for $r=2$ is obtained from the case $r=1$ by the mere substitution of $\mathcal{D}_2^*$ for $\mathcal{D}_1^*$; which fact is reflected in all the inequalities so far obtained by the replacement of k_n, wherever it occurs, by Lk_n^2. Thus:

$$| C^*_{2,n}(\omega^m\alpha) | \leqq L_1Lk_n^2; \qquad C^*_{2,n}(t) \ll \sigma MLk_n^2(1+t+\cdots+t^{k-1});$$

$$\Gamma_{2,n;j}(t) \ll \frac{QLk_n^2}{1-\theta''t}; \qquad B^*_{2,n} \ll \frac{QLk_n^2}{1-\theta''t}.$$

From (5.12), for $r=3$, we therefore have

$$D^*_{3,n}(t) = -\sum_{j=0}^{\infty}\{\beta_{1,n;j}t^jC^*_{2,n+j}(t) + \beta_{2,n;j}t^jC^*_{1,n+j}(t)\}$$

$$\ll \sum_{j=0}^{\infty} 2\sigma MQL(\theta''t)^j(1+t+\cdots+t^{k-1});$$

so

$$(5.43) \qquad D^*_{3,n}(t) \ll \frac{2\sigma MQL\zeta}{1-\theta''t}\, k_n^3 = \frac{2L^2k_n^3}{1-\theta''t}.$$

In general, a simple induction gives the majorants

$$D^*_{r,n}(t) \ll \frac{P_r L^{r-1} k_n^r}{1-\theta'' t}, \tag{5.44}$$

where the P_r's are numerical constants ($P_1=1$, $P_2=1$), and

$$\begin{aligned} C^*_{r,n}(t) &\ll \sigma M P_r L^{r-1} k_n^r (1 + t + \cdots + t^{k-1}), \\ B^*_{r,n}(t) &\ll \frac{Q P_r L^{r-1} k_n^r}{1-\theta'' t}. \end{aligned} \tag{5.45}$$

Hence, systems $\mathcal{B}_r^*$, $\mathcal{C}_r^*$, $\mathcal{D}_r^*$ are all admissible.

As for $\{P_r\}$, the induction that establishes (5.44) and (5.45) shows that

$$P_r = P_1 P_{r-1} + P_2 P_{r-2} + \cdots + P_{r-1} P_1 \qquad (r \geqq 2) \tag{5.46}$$

with $P_1 = P_2 = 1$.

LEMMA 5.4. *If*

$$\pi(t) = \sum_{j=0}^{\infty} P_{j+1} t^j \tag{5.47}$$

then

$$\pi(t) = \frac{1}{2t}\{1 - (1-4t)^{1/2}\}. \tag{5.48}$$

Consequently,

$$P_{j+1} = -2^{-1}(-4)^{j+1}\binom{1/2}{j+1} \tag{5.49}$$

and

$$|P_j| \leqq 4^j. \tag{5.50}$$

For, from (5.48) we have

$$t\pi^2(t) - \pi(t) + 1 = 0,$$

so (5.46) follows from (5.47). Since $\pi(0) = 1$, the coefficients in the expansion for $\pi(t)$ coincide completely with the sequence $\{P_{j+1}\}$ of (5.44)–(5.46). If (5.48) is expanded by the binomial theorem it is seen that P_{j+1} has the value (5.49). The inequality (5.50) now follows from the ratio

$$\frac{P_{j+1}}{P_j} = 4\left(\frac{2j-1}{2j+1}\right) < 4.$$

Let us now return to (5.10). Define $\mathcal{B}^*$, $\mathcal{C}^*$ by

$$\mathcal{B}^* = \sum_{r=1}^{\infty} \lambda^r \mathcal{B}_r^*, \qquad \mathcal{C}^* = \sum_{r=1}^{\infty} \lambda^r \mathcal{C}_r^*. \tag{5.51}$$

Then

$$\begin{aligned} B_n^*(t) &\equiv \sum_0^\infty \beta_{nj} t^j \ll Q\left(\sum_{r=1}^\infty \lambda^r P_r L^{r-1} k_n^r\right)(1-\theta'' t)^{-1}, \\ C_n^*(t) &\equiv \sum_0^\infty \gamma_{nj} t^j \ll \sigma M\left(\sum_{r=1}^\infty \lambda^r P_r L^{r-1} k_n^r\right)(1+t+\cdots+t^{k-1}). \end{aligned} \tag{5.52}$$

Since the majorants (5.52) are obtained by using absolute values everywhere in the determination of estimates for each $\mathcal{B}_r^*$, $\mathcal{C}_r^*$, it is clear that $\mathcal{B}^*$ and $\mathcal{C}^*$ are admissible, and that (5.10) is valid, for all λ for which

$$|\lambda| < \frac{1}{4Lk_0}. \tag{5.53}$$

(We may use k_0 since $k_n \downarrow 0$.)

To handle the system $\mathcal{A}_1+\mathcal{E}^*$ the choice $\lambda=1$ must be possible. This choice is permissible *if we assume* that

$$4Lk_0 < 1. \tag{5.54}$$

On checking back it will be found that L is determined by the system $\mathcal{A}_1$ (hence by $\mathcal{A}$) and by θ; *it is not affected by the values of the sequence* $\{k_n\}$.

We may sum up our results as follows:

LEMMA 5.5. *Let $\mathcal{A}_1+\mathcal{E}^*$ be admissible, with $\mathcal{A}$ of order $l>0$. Let the zeros of $\Delta_A(t)$ be $\{\omega^f\alpha_j\}$ ($f=0, 1, \cdots, k-1; j=1, \cdots, l$), and suppose that* (5.7) *holds*: $|\alpha_l|<|\alpha_j|$, $j=1, \cdots, l-1$. *If L, determined solely by $\mathcal{A}_1$ and by θ, satisfies condition* (5.54), *then $\mathcal{A}_1+\mathcal{E}^*$ has the factorization*

$$\mathcal{A}_1 + \mathcal{E}^* = (\mathcal{B} + \mathcal{B}^*)(\mathcal{C} + \mathcal{C}^*), \tag{5.55}$$

where $\mathcal{B}^*$, $\mathcal{C}^*$ are the admissible systems given by (5.51) *for* $\lambda=1$.

We should like to extend this factorization further. It is to be noted that the condition $|\alpha_l|<|\alpha_j|$ ($j=1, \cdots, l-1$) was imposed in order that we be assured that system (5.22) has for each m a solution $x_n^{(m)}$ that goes to zero as $n\to\infty$. (Theorem 4.5 was invoked for this purpose.) If therefore we are to factor $\mathcal{B}+\mathcal{B}^*$ (using the method that leads to Lemma 5.5), we need to assume that $|\alpha_{l-1}|<|\alpha_j|$ for $j<l-1$; and so on. We are thus led to the condition

$$|\alpha_l| < |\alpha_{l-1}| < \cdots < |\alpha_1|, \tag{5.56}$$

which we now assume.

From (5.9),

$$\mathcal{B} = \mathcal{C}_1\mathcal{B}_1 \cdots \mathcal{B}_{l-1}\mathcal{C}_l,$$

so

$$\mathcal{B} + \mathcal{B}^* = (\mathcal{C}_1\mathcal{B}_1 \cdots \mathcal{B}_{l-1} + \mathcal{B}^*\mathcal{C}_l^{-1})\mathcal{C}_l.$$

$\mathcal{C}_l$, of order zero, may be retained as it is; accordingly, it is the parenthesis that is to be factored. Now $\mathcal{B}^*\mathcal{C}_l^{-1}$ is by Lemma 5.1 an admissible system of perturbation terms, but we need to know something about the magnitude of its coefficients. This information is supplied by the following, easily proved, lemma:

LEMMA 5.6. *Let $\mathcal{C}$ and $\mathcal{A}^*$ be admissible. Let constants M, θ', k_n, θ be chosen (as they can be) so that*

$$C_n(t) \ll \frac{M}{1 - \theta' t}, \qquad A_n^*(t) \ll \frac{k_n}{1 - \theta t},$$

with $\theta' < 1/q$, $\theta < 1/q$, $k_n \downarrow 0$. Then M', θ'' exist independent of $\{k_n\}$, with $\theta'' < 1/q$, such that the admissible system $\mathcal{D}^* = \mathcal{C}\mathcal{A}^*$ satisfies the relations

$$D_n^*(t) \ll \frac{M' k_n}{1 - \theta'' t}.$$

If we apply Lemma 5.6 to $\mathcal{B}^*\mathcal{C}_l^{-1}$ we see that the factorization of

$$\mathcal{C}_1\mathcal{B}_1 \cdots \mathcal{B}_{l-1} + \mathcal{B}^*\mathcal{C}_l^{-1}$$

can be carried out in exactly the same manner as that of $\mathcal{A}_1 + \mathcal{E}^*$. The various constants that appear in the earlier majorant expressions (M, L_1, L, and so on) may have new values, but these values continue to be independent of $\{k_n\}$. Suppose we replace L in (5.54) by the notation $L^{(l)}$, to indicate that it arose when the zero $\alpha = \alpha_l$ was under consideration. Then in the present case we obtain an $L^{(l-1)}$; and so on down to $L^{(2)}$, at which point only a factor of order *one* is left, and this factor is essentially in final form, so the process ends.

We therefore have the following result:

THEOREM 5.1. *Let $\mathcal{A} + \mathcal{A}^*$ be admissible, where $\mathcal{A}$, of order $l > 0$, has the representation* (5.8). *Let the zeros of $\Delta_A(t)$ be $\{\omega^f \alpha_j\}$ $(f = 0, 1, \cdots, k-1;$ $j = 1, \cdots, l)$, and suppose* (5.56) *holds. Let* (5.54) *be satisfied by $L = L^{(j)}$, $j = l, l-1, \cdots, 2$. Then $\mathcal{A} + \mathcal{A}^*$ has the factorization*

$$(5.57) \quad \mathcal{A} + \mathcal{A}^* = \mathcal{C}_1(\mathcal{B}_1 + \mathcal{B}^{(1)*})\mathcal{C}_2(\mathcal{B}_2 + \mathcal{B}^{(2)*})\mathcal{C}_3 \cdots (\mathcal{B}_l + \mathcal{B}^{(l)*})\mathcal{C}_{l+1},$$

where the systems $\mathcal{B}^{(j)*}$ are admissible, and where the $\mathcal{C}$'s and $\mathcal{B}$'s have the properties stated in Theorem 4.4.

Theorem 5.1 raises two questions: What if (5.56) does not hold, and what if (5.54) fails to be true for $L = L^{(j)}$, $j = l, l-1, \cdots, 2$? In the former case, that is, when two or more nests of zeros of $\Delta_A(t)$ are equal or have the same magnitude, our methods do not supply a factorization. It would be of interest to obtain such a factorization if it is possible (as one may conjecture). In the latter case the failure of (5.54) can be circumvented by replacing system $\mathcal{A}+\mathcal{A}^*$ by a suitable one of its *truncates*, as described below.

DEFINITION. Given a system $\mathcal{H}:H_n[X]$, $n=0, 1, \cdots$. If $\{y_n\}$ is defined by

$$y_n = x_{n+p}$$

and $\{K_n(t)\}$ by

$$K_n(t) = H_{n+p}(t),$$

then $\mathcal{K}_p: K_n[Y]$, $n=0, 1, \cdots$, is called *the truncate of* $\mathcal{H}$ *of degree* p. The truncate $\mathcal{K}_p$ is thus seen to be obtained by removing the first p forms ($n=0, 1, \cdots, p-1$) of $\mathcal{H}$ and relabelling the x's.

If $h_{n0}\neq 0$, $n=0, 1, \cdots$, this replacement of a system $\mathcal{H}$ by a truncate system $\mathcal{K}$ does not alter the character of the system of equations

$$\mathcal{H}:\quad H_n[X] = c_n \qquad (n = 0, 1, \cdots).$$

For to every solution $\{x_n\}$ of this system corresponds the solution $\{y_n=x_{n+p}\}$ of

$$\mathcal{K}:\quad K_n[Y] = c_n' \qquad (c_n' = c_{n+p}),$$

and conversely.

In the case of an admissible system $\mathcal{A}+\mathcal{A}^*$, if we choose the degree p of the truncate system to be a multiple of $k: p=rk$, then the truncate has the form $\mathcal{A}+\mathcal{D}^*$, where system $\mathcal{A}$ is *unchanged*, and $D_n^*(t)=A_{n+rk}^*(t)$. Clearly $\mathcal{D}^*$ is admissible; hence so is $\mathcal{A}+\mathcal{D}^*$. We may choose r large enough so that

$$4L^{(j)}k_p < 1 \qquad (j = l, l-1, \cdots, 2). \tag{5.58}$$

On setting

$$k_n' = k_{n+rk} = k_{n+p} \tag{5.59}$$

for all n, then $k_n' \downarrow 0$ and

$$4L^{(j)}k_0' < 1 \qquad (j = l, l-1, \cdots, 2). \tag{5.60}$$

Thus, the truncate system $\mathcal{A}+\mathcal{D}^*$ does satisfy (5.54) for $L=L^{(j)}$ $(2\leqq j\leqq l)$. Since by hypothesis the admissible system $\mathcal{A}+\mathcal{A}^*$ satisfies (5.5), this is also true of $\mathcal{A}+\mathcal{D}^*$; hence as was pointed out the two systems of equations

$$(\mathcal{A}+\mathcal{A}^*)_n[X] = c_n; \qquad (\mathcal{A}+\mathcal{D}^*)_n[Y] = c_n' \qquad (c_n' = c_{n+p})$$

are equivalent. We proceed to the solution of the latter system.

REMARK. In solving a truncate system it is logically desirable to state in advance how far down in the original system the truncate begins; that is, to specify the degree of the truncate. However, this often requires the introduction of a considerable number of preliminary statements and definitions regarding new quantities, that would be unmotivated at the time of their appearance. To avoid this, we shall work with the original (untruncated) system; and as the argument progresses, the above-referred to new quantities will appear in a natural way. These quantities will dictate which truncate is to be used in place of the original system, and it will be seen that the arguments used will be valid for that truncate.

LEMMA 5.7. *Let $\mathcal{A}+\mathcal{A}^*$ be admissible (in $|t|\leqq q$), with $\mathcal{A}$ of order zero. Then the system of equations*

$$\mathcal{A}+\mathcal{A}^*:\quad (\mathcal{A}+\mathcal{A}^*)_n[X] = c_n \qquad (n = 0, 1, \cdots) \tag{5.61}$$

has a unique admissible solution for every sequence $\{c_n\}$ for which $((c_n))\leqq q$. In particular, if $c_n\equiv 0$ then $x_n\equiv 0$.

Let $((c_n))\leqq q$. Since $\mathcal{A}$ is of order zero, we may operate on (5.61) with $\mathcal{A}^{-1}$ to obtain the *equivalent system*

$$(\mathfrak{I}+\mathcal{A}^{**})_n[X] = c_n' \equiv (A^{-1})_n[\{c_j\}] \qquad (n = 0, 1, \cdots),$$

with $((c_n'))\leqq q$. Here $\mathfrak{I}$ is the identity and $\mathcal{A}^{**}=\mathcal{A}^{-1}\mathcal{A}^*$ is an admissible system of perturbation terms. In other words, we need only consider the system

$$x_n + \sum_{j=0}^{\infty} \alpha_{nj}x_{n+j} = c_n' \qquad (n = 0, 1, \cdots). \tag{5.62}$$

Our assumptions are such that $M=M(\epsilon)$ exists for which

$$|c_n'| \leqq M(\delta+\epsilon)^n, \tag{5.63}$$

where $((c_n))=\delta\leqq q$ and where $\epsilon>0$ is chosen small enough so that the functions $A_n^{**}(t)=\sum_0^\infty \alpha_{nj}t^j$ are analytic in a circle of radius greater than $\delta+\epsilon$, independent of n. Also,

$$|\alpha_{nj}| \leqq k_n\theta^j \qquad (k_n \downarrow 0;\ \theta < 1/q); \tag{5.64}$$

and ϵ is further restricted so that

$$\theta(q+\epsilon) < 1.$$

We now apply the method of successive approximations, defining $x_n^{(r)}$ by

$$\begin{aligned} x_n^{(1)} &= c_n'; \\ x_n^{(r)} &= c_n' - \sum_{j=0}^{\infty} \alpha_{nj}x_{n+j}^{(r-1)} \qquad (r = 2, 3 \cdots). \end{aligned} \tag{5.65}$$

A simple induction leads to

$$(5.66)\qquad \left| x_n^{(r)} \right| \leqq M(\delta + \epsilon)^n T_{n,r}$$

where

$$T_{n,r} = \sum_{p=0}^{r-1} \left\{ \frac{k_n}{1 - \theta(\delta + \epsilon)} \right\}^p.$$

It is no restriction to assume, as we do, that

$$(5.67)\qquad \beta \equiv \frac{k_0}{1 - \theta(\delta + \epsilon)} < 1,$$

since by suitable truncation of (5.62) an *equivalent* system can be obtained for which, in terms of the *new* sequence $\{k_n\}$, (5.67) does hold. (See the remark that precedes Lemma 5.7.)

From the relations

$$(5.68)\quad x_n^{(2)} - x_n^{(1)} = -\sum_{j=0}^{\infty} \alpha_{nj} x_{n+j}^{(1)}; \quad x_n^{(r)} - x_n^{(r-1)} = -\sum_{j=0}^{\infty} \alpha_{nj} \left\{ x_{n+j}^{(r-1)} - x_{n+j}^{(r-2)} \right\}$$

we obtain (again by a simple induction)

$$(5.69)\quad \left| x_n^{(r)} - x_n^{(r-1)} \right| \leqq M(\delta + \epsilon)^n \left\{ \frac{k_n}{1 - \theta(\delta + \epsilon)} \right\}^{r-1} \leqq M\beta^{r-1}(\delta + \epsilon)^n.$$

Since $\beta < 1$ we deduce the existence of

$$(5.70)\qquad x_n \equiv \lim_{r \to \infty} x_n^{(r)} \qquad (n = 0, 1, \cdots).$$

Moreover, since $T_{n,r} \leqq 1/(1-\beta)$ for all n and r, we see from (5.66) that

$$\left| x_n^{(r)} \right| \leqq \frac{M}{1 - \beta} (\delta + \epsilon)^n;$$

so

$$| x_n | \leqq \frac{M}{1 - \beta} (\delta + \epsilon)^n, \qquad (n = 0, 1, \cdots).$$

From the arbitrariness of ϵ we conclude that $((x_n)) \leqq \delta \leqq q$. The sequence $\{x_n\}$ is therefore admissible.

There remains to show that $\{x_n\}$ is a solution of (5.62), and that this solution is unique. From (5.69),

$$\left| x_n^{(p+r)} - x_n^{(r)} \right| \leqq M(\delta + \epsilon)^n \cdot \sum_{j=r}^{r+p-1} \beta^j;$$

and on letting $p \to \infty$:

$$(5.71)\qquad | x_n - x_n^{(r)} | \leqq \frac{M\beta^r}{1-\beta}(\delta+\epsilon)^n.$$

Now

$$\begin{aligned} \left| x_n - c_n' + \sum_{j=0}^{\infty} \alpha_{nj}x_{n+j} \right| &= \left| x_n - x_n^{(r)} + \sum_{j=0}^{\infty} \alpha_{nj}\left\{ x_{n+j} - x_{n+j}^{(r-1)} \right\} \right| \\ &\leqq | x_n - x_n^{(r)} | + \sum_{j=0}^{\infty} | \alpha_{nj} | \cdot | x_{n+j} - x_{n+j}^{(r-1)} | \\ &\leqq \frac{M\beta^{r-1}}{1-\beta}(\delta+\epsilon)^n \left\{ \beta + k_n \sum_{j=0}^{\infty} [\theta(\delta+\epsilon)]^j \right\} \\ &< \frac{2M\beta^r}{1-\beta}(\delta+\epsilon)^n. \end{aligned}$$

As $r\rightarrow\infty$ the right side approaches zero for each fixed n; and since the left side is independent of r, that side *is* zero for each n. Thus it has been shown that $\{x_n\}$ given by (5.70) is an admissible solution of (5.62).

It is, moreover, unique. For suppose $\{y_n\}$, with $((y_n))\leqq q$, is a solution. A number λ exists for which

$$(5.72)\qquad \lambda \equiv \max \left| \frac{x_n - y_n}{(q+\epsilon)^n} \right| \qquad (n = 0, 1, \cdots).$$

(The value λ is actually attained for at least one n.) From the fact that $\{x_n\}$, $\{y_n\}$ are both solutions follows the relation

$$x_n - y_n = - \sum_{j=0}^{\infty} \alpha_{nj}(x_{n+j} - y_{n+j}),$$

so

$$| x_n - y_n | \leqq \lambda k_n \sum_{j=0}^{\infty} \theta^j (q+\epsilon)^{n+j} \leqq \lambda\beta(q+\epsilon)^n.$$

But $\beta<1$. Thus property (5.72), which makes λ minimal, is contradicted unless $\lambda=0$. Therefore $y_n\equiv x_n$, as was to be shown.

LEMMA 5.8. *Let $\mathcal{A}+\mathcal{A}^*$ be admissible (in $|t|\leqq q$), with $\mathcal{A}$ of order one. Then the system of equations*

$$(5.73)\qquad \mathcal{A}+\mathcal{A}^*:\quad (\mathcal{A}+\mathcal{A}^*)_n[X] = c_n \qquad (n = 0, 1, \cdots)$$

has an admissible solution $\{x_n\}$ for every $\{c_n\}$ for which $((c_n))\leqq q$. There exists a number R such that there is a unique admissible solution with the value x_R prescribed.

We can write $\mathcal{A}=\mathcal{C}_1\mathcal{B}\mathcal{C}_2$, where $\mathcal{C}_1$, $\mathcal{C}_2$ are admissible of order zero and $\mathcal{B}$ is canonical (corresponding, say, to the zero α and the index s):

$$B_s(t) = t^k - \alpha^k; \qquad B_j(t) = 1 \qquad (j \neq s; j = 0, 1, \cdots, k-1).$$

Applying $\mathcal{C}_1^{-1}$ on the left of (5.73) we obtain the equivalent system

$$(5.74) \quad (\{\mathcal{B} + \mathcal{A}^{**}\}\mathcal{C}_2)_n[X] = c_n' \equiv (\mathcal{C}_1^{-1})_n[\{c_j\}] \qquad (n = 0, 1, \cdots),$$

where $\mathcal{A}^{**}=\mathcal{C}_1^{-1}\mathcal{A}^*\mathcal{C}_2^{-1}$ is an admissible system of perturbation terms.

Consider the system

$$(5.75) \qquad (\mathcal{B} + \mathcal{A}^{**})_n[U] = c_n' \qquad (n = 0, 1, \cdots).$$

Suppose the lemma can be established for (5.75). Let $\{u_n^{(1)}\}$ be a particular solution, and let $\{u_n^{(0)}\not\equiv 0\}$ be a solution of the homogeneous equation, so that the general admissible solution of (5.75) is $u_n=u_n^{(1)}+\lambda u_n^{(0)}$ (λ arbitrary). Then the general solution of (5.74) is given by

$$x_n = x_n^{(1)} + \lambda x_n^{(0)}$$

where $\{x_n^{(1)}\}$, $\{x_n^{(0)}\}$ are the unique solutions of the respective equations

$$(\mathcal{C}_2)_n[X^{(1)}] = u_n^{(1)}; \qquad (\mathcal{C}_2)_n[X^{(0)}] = u_n^{(0)},$$

guaranteed by Theorem 2.5. By that theorem, if we denote the matrix of coefficients defined by $\mathcal{C}_2^{-1}$ by $\{g_{nj}\}$, we have

$$x_n = \sum_{j=0}^{\infty} g_{nj}u_{n+j}^{(1)} + \lambda \sum_{j=0}^{\infty} g_{nj}u_{n+j}^{(0)} = x_n^{(1)} + \lambda x_n^{(0)}.$$

Now $x_n^{(0)}\not\equiv 0$ else we would have $u_n^{(0)}\equiv 0$, contrary to the choice of $\{u_n^{(0)}\}$. If therefore R is chosen so that $x_R^{(0)}\neq 0$, then λ can be determined so that x_R has a prescribed value. Moreover λ will have a unique value, so the solution $\{x_n\}$ will be uniquely defined by the value x_R. In other words, if the lemma is true for (5.75) then it is true for (5.74).

We need therefore only consider system (5.75), which in expanded form can be written

$$(5.76) \quad \begin{aligned} x_{nk+i} &= c'_{nk+i} - \sum_{j=0}^{\infty} \alpha_{nk+i,j}x_{nk+i+j} \qquad (i \neq s;\ i = 0, 1, \cdots, k-1); \\ -\alpha^k x_{nk+s} + x_{(n+1)k+s} &= c'_{nk+s} - \sum_{j=0}^{\infty} \alpha_{nk+s,j}x_{nk+s+j} \quad (n = 0, 1, \cdots). \end{aligned}$$

As in Lemma 5.7, relations (5.63) and (5.64) hold, and we again set up a sequence of approximations:

$$\overset{(1)}{x}_{nk+i} = c'_{nk+i}; \qquad -\alpha^{k}\overset{(1)}{x}_{nk+s} + \overset{(1)}{x}_{(n+1)k+s} = c'_{nk+s};$$

(5.77)
$$\overset{(r)}{x}_{nk+i} = c'_{nk+i} - \sum_{j=0}^{\infty} \alpha_{nk+i,j}\overset{(r-1)}{x}_{nk+i+j};$$

$$-\alpha^{k}\overset{(r)}{x}_{nk+s} + \overset{(r)}{x}_{(n+1)k+s} = c'_{nk+s} - \sum_{j=0}^{\infty} \alpha_{nk+s,j}\overset{(r-1)}{x}_{nk+s+j}, \qquad (r = 2, 3, \cdots).$$

Let value x_s be assigned, and as we proceed in the proof we agree always to choose $x_s^{(r)} = x_s$. That such choice is possible will be seen. The second relation on the first line of (5.77) has the solution (compare §3, (3.37)–(3.40))

$$\overset{(1)}{x}_{nk+s} = \alpha^{nk}\overset{(1)}{x}_{s} + \sum_{j=0}^{n-1} \alpha^{jk} c'_{s+(n-j-1)k}.$$

$x_s^{(1)}$ can be chosen arbitrarily, but as was stated earlier we are taking $x_s^{(1)} = x_s$. From the inequalities

$$|c'_n| \leqq M(q+\epsilon)^n, \qquad |\alpha| < q+\epsilon,$$

we readily find that

$$|\overset{(1)}{x}_{nk+i}| \leqq M(q+\epsilon)^{nk+i},$$

$$|\overset{(1)}{x}_{nk+s}| \leqq (q+\epsilon)^{nk+s}\left\{\frac{|x_s|}{(q+\epsilon)^s} + \frac{M}{(q+\epsilon)^k - |\alpha|^k}\right\};$$

so that on setting

(5.78)
$$M_1 \equiv \max\left\{M, \frac{|x_s|}{(q+\epsilon)^s} + \frac{M}{(q+\epsilon)^k - |\alpha|^k}\right\},$$

then for *all* n,

(5.79)
$$|\overset{(1)}{x}_{n}| \leqq M_1(q+\epsilon)^n.$$

To continue the proof we make use of the following lemma.

LEMMA 5.9. *If* $z_n \to 0$ *and* $0 < \theta < 1$, *then*

$$\lim_{n\to\infty}\{z_n + z_{n-1}\theta + \cdots + z_0\theta^n\} = 0.$$

Denote the brace by t_n. The sequence $\{z_n\}$ is bounded: $|z_n| \leqq M$. Let $\epsilon > 0$ be given, and choose $N = N(\epsilon)$ so that $|z_n| < \epsilon$ for all $n > N$. Then

$$|t_n| \leqq M\sum_{j=0}^{N}\theta^{n-j} + \epsilon\sum_{j=N+1}^{n}\theta^{n-j} \leqq \frac{M\theta^{n-N} + \epsilon}{1-\theta},$$

so $t_n \to 0$.

Now turn back to (5.77). Using (5.79) we find that

$$\left| x_{nk+i}^{(2)} \right| = \left| c'_{nk+i} - \sum_{j=0}^{\infty} \alpha_{nk+i,j} x_{nk+i+j}^{(1)} \right| \leqq (q+\epsilon)^{nk+i} M_1 \left\{ 1 + \frac{k_{nk+i}}{1-\theta(q+\epsilon)} \right\};$$

$$\begin{aligned} \left| x_{nk+s}^{(2)} \right| &= \left| \alpha^{nk} x_s^{(2)} + \sum_{r=0}^{n-1} \alpha^{rk} \left\{ c'_{s+(n-r-1)k} - \sum_{j=0}^{\infty} \alpha_{nk+s-(r+1)k,j} x_{nk+s-(r+1)k+j}^{(1)} \right\} \right| \\ &\leqq |\alpha|^{nk} |x_s| \\ &\quad + M_1 (q+\epsilon)^{s+(n-1)k} \sum_{r=0}^{n-1} \left(\frac{|\alpha|}{q+\epsilon} \right)^{rk} \left\{ \frac{M}{M_1} + \frac{k_{nk+s-(r+1)k}}{1-\theta(q+\epsilon)} \right\}. \end{aligned}$$

Since $|\alpha|/(q+\epsilon)<1$ and $k_n \to 0$, Lemma 5.9 applies to the sum

$$t_{n-1} \equiv \sum_{r=0}^{n-1} \left(\frac{|\alpha|}{q+\epsilon} \right)^{rk} k_{nk+s-(r+1)k}. \tag{5.80}$$

Accordingly, $t_n \to 0$; and it is no restriction to suppose that $t_n \downarrow 0$ (since $\{t_n\}$ can be replaced if necessary by $\{t'_n\}$ where $t'_n \downarrow 0$ and $t_n \leqq t'_n$).

We thus have

$$\left| x_{nk+s}^{(2)} \right| \leqq M_1 (q+\epsilon)^{s+nk} \left\{ 1 + \frac{t_{n-1}}{(q+\epsilon)^k [1-\theta(q+\epsilon)]} \right\}.$$

Let

$$\lambda_n = \max \left\{ \frac{t_{n-1}}{(q+\epsilon)^k [1-\theta(q+\epsilon)]};\ \frac{k_{nk+j}}{1-\theta(q+\epsilon)} \right\} \ (j=0, 1, \cdots, k-1). \tag{5.81}$$

As $n \to \infty$, $\lambda_n \downarrow 0$ since this is true of t_n and k_n. We now have for all n and *all* $j=0, 1, \cdots, k-1$, $|x_{nk+j}^{(2)}| \leqq M_1(1+\lambda_n)(q+\epsilon)^{nk+j}$.

It is now a straightforward induction to obtain

$$\left| x_{nk+j}^{(r)} \right| \leqq M_1 (1 + \lambda_n + \cdots + \lambda_n^{r-1}) (q+\epsilon)^{nk+j}, \tag{5.82}$$

$j=0, 1, \cdots, k-1$; $n=0, 1, \cdots$; $r=1, 2, \cdots$; or, since $\lambda_n \downarrow 0$,

$$\left| x_n^{(r)} \right| \leqq M_1 (1 + \lambda_0 + \cdots + \lambda_0^{r-1}) (q+\epsilon)^n. \tag{5.83}$$

We conclude from the arbitrariness of ϵ that $((x_n^{(r)})) \leqq q$ for each r.

From (5.77) we obtain (for $r \geqq 2$)

$$\begin{aligned} x_{nk+i}^{(r)} - x_{nk+i}^{(r-1)} &= - \sum_{j=0}^{\infty} \alpha_{nk+i,j} \left\{ x_{nk+i+j}^{(r-1)} - x_{nk+i+j}^{(r-2)} \right\}, \\ - \alpha^k \left\{ x_{nk+s}^{(r)} - x_{nk+s}^{(r-1)} \right\} &+ \left\{ x_{(n+1)k+s}^{(r)} - x_{(n+1)k+s}^{(r-1)} \right\} \\ &= - \sum_{j=0}^{\infty} \alpha_{nk+s,j} \left\{ x_{nk+s+j}^{(r-1)} - x_{nk+s+j}^{(r-2)} \right\}. \end{aligned} \tag{5.84}$$

Here $i \neq s$; $i=0, 1, \cdots, k-1$; and $x_n^{(0)} \equiv 0$. The second of these relations can be solved (again as in §3, (3.37)–(3.40)) to give

$$\overset{(r)}{x}_{nk+s} - \overset{(r-1)}{x}_{nk+s} = \alpha^{nk}\left\{\overset{(r)}{x}_s - \overset{(r-1)}{x}_s\right\}$$
$$+ \sum_{p=0}^{n-1}\alpha^{pk}\left[-\sum_{j=0}^{\infty}\alpha_{(n-p-1)k+s,j}\cdot\left\{\overset{(r-1)}{x}_{(n-p-1)k+s+j} - \overset{(r-2)}{x}_{(n-p-1)k+s+j}\right\}\right];$$

or, since $\overset{(r)}{x}_s = \overset{(r-1)}{x}_s = x_s$:

$$\overset{(r)}{x}_{nk+s} - \overset{(r-1)}{x}_{nk+s} = -\sum_{p=0}^{n-1}\alpha^{pk}\sum_{j=0}^{\infty}\alpha_{(n-p-1)k+s,j}\left\{\overset{(r-1)}{x}_{(n-p-1)k+s+j} - \overset{(r-2)}{x}_{(n-p-1)k+s+j}\right\}.$$

It is readily found that

$$\left|\overset{(2)}{x}_{nk+i} - \overset{(1)}{x}_{nk+i}\right| \leqq M_1\lambda_0(q+\epsilon)^{nk+i}, \qquad \left|\overset{(2)}{x}_{nk+s} - \overset{(1)}{x}_{nk+s}\right| \leqq M_1\lambda_0(q+\epsilon)^{nk+s},$$

so

$$\left|\overset{(2)}{x}_n - \overset{(1)}{x}_n\right| \leqq M_1\lambda_0(q+\epsilon)^n;$$

and two simple inductions (one for i, the other for s) give the general inequality

$$(5.85) \qquad \left|\overset{(r)}{x}_n - \overset{(r-1)}{x}_n\right| \leqq M_1\lambda_0^{r-1}(q+\epsilon)^n \qquad (n = 0, 1, \cdots).$$

We now suppose that $\lambda_0 < 1$. This is no restriction, for if $\lambda_0 \geqq 1$, then since $\lambda_n \to 0$ there is an N such that $\lambda_n < 1$ for all $n > N$. Consequently by applying a suitable truncation, a system equivalent to (5.76) can be obtained, for which the new sequence $\{\lambda_n' \equiv \lambda_{n+p};\ p$ sufficiently large$\}$ has the above property: $\lambda_0' < 1$.

It follows from (5.85) that the limits

$$(5.86) \qquad x_n \equiv \lim_{r\to\infty} \overset{(r)}{x}_n \qquad (n = 0, 1, \cdots)$$

exist, and from (5.83) that

$$(5.87) \qquad |x_n| \leqq \frac{M_1}{1-\lambda_0}(q+\epsilon)^n,$$

so that $((x_n)) \leqq q$. In other words, $\{x_n\}$ is admissible. Also, since $\overset{(r)}{x}_s = x_s$ (whose value is assigned), we see that $\{x_n\}$ has for $n = s$ that same assigned value.

We need to show that $\{x_n\}$ is a solution of (5.76), and that it is unique for the prescribed value x_s. From (5.85) we obtain

$$\left|\overset{(p+r)}{x}_n - \overset{(r)}{x}_n\right| \leqq M_1\lambda_0^r(1 + \lambda_0 + \cdots + \lambda_0^{p-1})(q+\epsilon)^n,$$

so letting $p \to \infty$:

$$| x_n - x_n^{(r)} | \leqq \frac{M_1\lambda_0^r}{1-\lambda_0}(q+\epsilon)^n.$$

Let $i=0, 1, \cdots, k-1$ but $i \neq s$. From (5.77) we have

$$\left| x_{nk+i} - c'_{nk+i} + \sum_{j=0}^{\infty} \alpha_{nk+i,j}x_{nk+i+j} \right|$$

$$\leqq | x_{nk+i} - x_{nk+i}^{(r)} | + \sum_{j=0}^{\infty} k_{nk+i}\theta^j | x_{nk+i+j} - x_{nk+i+j}^{(r-1)} | \leqq \frac{2M_1}{1-\lambda_0}\lambda_0^r(q+\epsilon)^{nk+i}.$$

The right side goes to zero as $r \to \infty$; hence the left side *is* zero for each n. Thus part of system (5.76) is satisfied by $\{x_n\}$. Again from (5.77),

$$\left| -\alpha^k x_{nk+s} + x_{(n+1)k+s} - c'_{nk+s} + \sum_{j=0}^{\infty} \alpha_{nk+s,j}x_{nk+s+j} \right|$$

$$= \Big| -\alpha^k\{x_{nk+s} - x_{nk+s}^{(r)}\} + \{x_{(n+1)k+s} - x_{(n+1)k+s}^{(r)}\}$$
$$+ \sum_{j=0}^{\infty} \alpha_{nk+s,j}\{x_{nk+s+j} - x_{nk+s+j}^{(r-1)}\} \Big|$$
$$\leqq \frac{M_1}{1-\lambda_0}\lambda_0^r(q+\epsilon)^{nk+s}\{ |\alpha|^k + (q+\epsilon)^k + 1\};$$

and again the right side approaches zero as $r \to \infty$, so the left side *is* zero. This gives us the rest of system (5.76): $\{x_n\}$ is an admissible solution of (5.76).

To establish uniqueness, suppose $\{y_n\}$ is an admissible solution for which $y_s = x_s$. Then

$$x_{nk+i} - y_{nk+i} = -\sum_{j=0}^{\infty} \alpha_{nk+i,j}\{x_{nk+i+j} - y_{nk+i+j}\};$$

$$-\alpha^k\{x_{nk+s} - y_{nk+s}\} + \{x_{(n+1)k+s} - y_{(n+1)k+s}\}$$
$$= -\sum_{j=0}^{\infty} \alpha_{nk+s,j}\{x_{nk+s+j} - y_{nk+s+j}\}.$$

Since $((x_n)) \leqq q$, $((y_n)) \leqq q$, there exists a number μ such that

$$\mu \equiv \max | (x_n - y_n)/(q+\epsilon)^n |. \tag{5.88}$$

We find then that

$$| x_{nk+i} - y_{nk+i} | \leqq \sum_{j=0}^{\infty} k_{nk+i}\theta^j\mu(q+\epsilon)^{nk+i+j} \leqq \mu\lambda_0(q+\epsilon)^{nk+i}.$$

Also, as in (3.37)–(3.40):

$$x_{nk+s} - y_{nk+s} = \sum_{p=0}^{n-1} \alpha^{pk} \left[- \sum_{j=0}^{\infty} \alpha_{(n-p-1)k+s,j} \{ x_{(n-p-1)k+s+j} - y_{(n-p-1)k+s+j} \} \right];$$

so

$$| x_{nk+s} - y_{nk+s} | \leqq \mu(q+\epsilon)^{nk+s} \cdot \frac{t_{n-1}}{(q+\epsilon)^k[1-\theta(q+\epsilon)]} \leqq \mu\lambda_0(q+\epsilon)^{nk+s}.$$

Thus for all n,

$$| x_n - y_n | \leqq \mu\lambda_0(q+\epsilon)^n.$$

But $\lambda_0 < 1$, so (5.88) shows that $\mu = 0$. That is, $y_n \equiv x_n$. The lemma is now completely proved.

REMARK. If $\lambda_0 < 1$ in the original system the number R of Lemma 5.8 can be chosen as s. (And s, be it noted, can be made to have the value zero.) However, if truncation is necessary, R will be larger.

THEOREM 5.2. *Let $\mathcal{A}+\mathcal{A}^*$ be admissible (in $|t| \leqq q$), with $\mathcal{A}$ of order l and with condition* (5.56) *holding; and let $\{c_n\}$ be of type not exceeding q. The homogeneous system of equations*

$$(5.89) \qquad (\mathcal{A} + \mathcal{A}^*)_n[X] = 0$$

has precisely l linearly independent admissible solutions; and the corresponding nonhomogeneous system (with $\{0\}$ on the right replaced by $\{c_n\}$) has an admissible solution. The most general admissible solution of this latter system contains exactly l independent arbitrary constants.

It is no restriction to assume that system $\mathcal{A}+\mathcal{A}^*$ fulfills the conditions of Theorem 5.1, since these conditions can be achieved by a suitable truncation. Accordingly, we may suppose that $\mathcal{A}+\mathcal{A}^*$ has the factorization (5.57). Each $\mathcal{C}_j$ is of order zero, and each $\mathcal{B}_j$ of order one and canonical, so Lemmas 5.7 and 5.8 can be applied, together with the extension of Lemma 3.1 to cover more factors (as was done in Theorem 3.6). From this follows the assertion of Theorem 5.2.

6. **Perturbation system, general case.** Although we have not found a factorization for the general case of a perturbation system, it is possible to give a complete treatment of the corresponding system of equations, and this we do in the present section. This treatment is to be regarded as supplementing the work of §5 rather than superseding it, and for two reasons: (i) When applicable, the solution of the system of equations provided in §5 is less complex than that of the present section. (ii) The problem of factorization is of interest in itself, and is as desirable of achievement as is the problem of solving the related system of equations.

As a preliminary, we go back to the case of the k-periodic admissible system

$$\mathcal{A}_n[X] = \gamma_n \qquad (((\gamma_n)) \leqq q), \tag{6.1}$$

which was treated in §3 (Theorem 3.6). Now, however, we shall use the decomposition of $\mathcal{A}$ found in Theorem 4.4:

$$\mathcal{A} = \mathcal{C}_1\mathcal{B}_1\mathcal{C}_2\mathcal{B}_2 \cdots \mathcal{C}_l\mathcal{B}_l\mathcal{C}_{l+1}, \tag{6.2}$$

where the $\mathcal{C}_j$'s are of order zero and the $\mathcal{B}_j$'s are canonical corresponding to prescribed integers $s_1, \cdots, s_l$. We take all s_j's equal to a fixed integer s.

If we define $\{\gamma_n'\}$ by

$$\gamma_n' = (\mathcal{C}_1^{-1})_n[\{\gamma_j\}],$$

(6.1) is seen to be equivalent to

$$(\mathcal{C}_1^{-1}\mathcal{A})_n[X] = \gamma_n';$$

and in order to avoid complication of notation we may suppose, with no loss of generality, that $\mathcal{C}_1 = \mathcal{C}_1^{-1}$ is the identity. That is, we may suppose that (6.2) reduces to

$$\mathcal{A} = \mathcal{B}_1\mathcal{C}_2\mathcal{B}_2 \cdots \mathcal{B}_l\mathcal{C}_{l+1}. \tag{6.3}$$

Then (6.1) becomes

$$(\mathcal{B}_1\mathcal{C}_2 \cdots \mathcal{B}_l\mathcal{C}_{l+1})_n[X] = \gamma_n. \tag{6.4}$$

Let

$$(\mathcal{B}_1)_n[{}^{(1)}U] = \gamma_n; \tag{6.5}$$

then

$$(\mathcal{C}_2 \cdots \mathcal{B}_l\mathcal{C}_{l+1})_n[X] = {}^{(1)}U.$$

As we know from §3, (3.37)–(3.40), system (6.5) has the general solution

$$\begin{aligned} {}^{(1)}u_r &= \gamma_r \qquad\qquad r \not\equiv s \pmod{k}, \\ {}^{(1)}u_{s+nk} &= \alpha_1^{nk} \cdot \lambda_1 + \sum_{j=0}^{n-1} \alpha_1^{jk}\gamma_{s+(n-j-1)k}, \end{aligned}$$

with ${}^{(1)}u_s = \lambda_1$ arbitrary.

Let ${}^{(1)}U_0: \{{}^{(1)}u_{0,n}\}$, ${}^{(1)}U_1: \{{}^{(1)}u_{1,n}\}$ be defined by

$${}^{(1)}u_{0,r} = 0,\ r \not\equiv s;\quad {}^{(1)}u_{0,s+nk} = \alpha_1^{nk} \quad (n = 0, 1, \cdots);$$

$${}^{(1)}u_{1,r} = \gamma_r,\ r \not\equiv s;\quad {}^{(1)}u_{1,s+nk} = \sum_{j=0}^{n-1} \alpha_1^{jk}\gamma_{s+(n-j-1)k} \quad (n = 0, 1, \cdots).$$

Then

$$^{(1)}U = {}^{(1)}U_1 + \lambda_1 {}^{(1)}U_0.$$

Consequently,

$$(\mathcal{C}_2 \cdots \mathcal{B}_l \mathcal{C}_{l+1})_n[X] = {}^{(1)}U_1 + \lambda_1 {}^{(1)}U_0.$$

Let ${}^{(1)}V_1$, ${}^{(1)}V_0$ be the unique solutions of the equations

$$(\mathcal{C}_2)_n[{}^{(1)}V_i] = {}^{(1)}U_i, \qquad i = 1, 2,$$

and define ${}^{(1)}V$ by

$$^{(1)}V = {}^{(1)}V_1 + \lambda_1 {}^{(1)}V_0,$$

so that

$$(\mathcal{C}_2)_n[{}^{(1)}V] = {}^{(1)}U = {}^{(1)}U_1 + \lambda_1 {}^{(1)}U_0.$$

Then

$$(\mathcal{B}_2 \cdots \mathcal{B}_l \mathcal{C}_{l+1})_n[X] = {}^{(1)}V_1 + \lambda_1 {}^{(1)}V_0.$$

Observe that ${}^{(1)}U_0$ and ${}^{(1)}V_0$ are independent of the sequence $\{\gamma_n\}$.

Let ${}^{(2)}U$ be a solution of

$$(6.6) \qquad (\mathcal{B}_2)_n[{}^{(2)}U] = {}^{(1)}V_1 + \lambda_1 {}^{(1)}V_0.$$

Then

$$(\mathcal{C}_3 \cdots \mathcal{B}_l \mathcal{C}_{l+1})_n[X] = {}^{(2)}U.$$

System (6.6) can be solved as was (6.5): the most general ${}^{(2)}U$ has the form

$$^{(2)}U = {}^{(2)}U_2 + \lambda_1 {}^{(2)}U_1 + \lambda_2 {}^{(2)}U_0,$$

where

$$\overset{(2)}{u}_{0,r} = 0, \quad r \neq s; \qquad \overset{(2)}{u}_{1,r} = \overset{(1)}{v}_{0,r};$$

$$\overset{(2)}{u}_{0,s+nk} = \alpha_2^{nk}; \qquad \overset{(2)}{u}_{1,s+nk} = \sum_{j=0}^{n-1} \alpha_2^{jk} \cdot \overset{(1)}{v}_{0,s+(n-j-1)k};$$

$$\overset{(2)}{u}_{2,r} = \overset{(1)}{v}_{1,r};$$

$$\overset{(2)}{u}_{2,s+nk} = \sum_{j=0}^{n-1} \alpha_2^{jk} \cdot \overset{(1)}{v}_{1,s+(n-j-1)k}.$$

Also, if we define ${}^{(2)}V_i$, $i=0, 1, 2$, as the unique solutions of

$$(\mathcal{C}_3)_n[{}^{(2)}V_i] = {}^{(2)}U_i,$$

then

$$^{(2)}V = {}^{(2)}V_2 + \lambda_1 {}^{(2)}V_1 + \lambda_2 {}^{(2)}V_0$$

satisfies $(\mathcal{C}_3)_n[{}^{(2)}V]={}^{(2)}U$. Here also, ${}^{(2)}V_1$, ${}^{(2)}V_0$ are independent of $\{\gamma_n\}$.

This process can be continued until we get

$$(6.7)\qquad {}^{(l)}U = {}^{(l)}U_l + \lambda_1\,{}^{(l)}U_{l-1} + \cdots + \lambda_l\,{}^{(l)}U_0$$

as the general solution of

$$(6.8)\qquad (\mathcal{B}_l)_n[{}^{(l)}U] = {}^{(l-1)}V \equiv {}^{(l-1)}V_{l-1} + \lambda_1\,{}^{(l-1)}V_{l-2} + \cdots + \lambda_{l-1}\,{}^{(l-1)}V_0,$$

and

$$(6.9)\qquad {}^{(l)}V = {}^{(l)}V_l + \lambda_1\,{}^{(l)}V_{l-1} + \cdots + \lambda_l\,{}^{(l)}V_0$$

as the unique solution of

$$(6.10)\qquad (\mathcal{C}_{l+1})_n[{}^{(l)}V] = {}^{(l)}U.$$

But

$$(\mathcal{C}_{l+1})_n[X] = {}^{(l)}U,$$

so

$$X = {}^{(l)}V.$$

To sum up, we have the following lemma.

LEMMA 6.1. *Let $\mathcal{A}$ be admissible of order $l>0$, and let $\{c_n\}$ be admissible. Define $\{\gamma_n\}$ by $\gamma_n=(\mathcal{C}_1^{-1})_n[\{c_j\}]$. Then the most general admissible solution of*

$$(6.11)\qquad (\mathcal{A})_n[X] = c_n \qquad (n = 0, 1, \cdots)$$

is given by

$$(6.12)\qquad X = {}^{(l)}V_l + \sum_{p=1}^{l} \lambda_p\,{}^{(l)}V_{l-p},$$

where the λ's are arbitrary constants. The sequences ${}^{(l)}V_i$ $(i=0, 1, \cdots, l-1)$ are independent of $\{c_n\}$ and $\{\gamma_n\}$.

If $c_n\equiv 0$, then $\gamma_n\equiv 0$, so ${}^{(l)}V_l$ is then also the zero sequence. The sequences ${}^{(l)}V_i$ $(i<l)$ are therefore solutions of the homogeneous system $(\mathcal{A})_n[X]=0$ $(n=0, 1, \cdots)$, and from Theorem 3.6 they are *linearly independent.*

We now pass to the admissible perturbation system (1.6), which we write as

$$(6.13)\qquad (\mathcal{A} + \mathcal{A}^*)_n[X] = c_n \qquad (n = 0, 1, \cdots).$$

It is no restriction to suppose, as we do, that $\mathcal{A}$ has the form (6.3). For if $\mathcal{A}$ is given by (6.2), then

$$\mathcal{A} + \mathcal{A}^* = \mathcal{C}_1(\mathcal{B}_1\mathcal{C}_2 \cdots \mathcal{B}_l\mathcal{C}_{l+1} + \mathcal{C}_1^{-1}\mathcal{A}^*),$$

and since $\mathcal{A}^*$ is an admissible system of perturbation terms, the same is true of $\mathcal{C}_1^{-1}\mathcal{A}^*$ by Lemma 5.1. Thus (6.13) is equivalent to

$$(\mathcal{C}_1^{-1}\mathcal{A} + \mathcal{C}_1^{-1}\mathcal{A}^*)_n[X] = c_n' \equiv (\mathcal{C}_1^{-1})_n[\{c_j\}];$$

and $\mathcal{C}_1^{-1}\mathcal{A}$ has the desired form (6.3).

LEMMA 6.2. *Let $\mathcal{B}$ be of order one and canonical (in $|t| \leqq q$), corresponding to the integer s and the zero α. Let $\epsilon > 0$ be given. If $|z_n| \leqq M(q+\epsilon)^n$, then the solution U of*

$$(\mathcal{B})_n[U] = z_n \qquad (n = 0, 1, \cdots)$$

given by

$$u_r = z_r, \qquad r \not\equiv s$$

$$u_{s+nk} = \sum_{j=0}^{n-1} \alpha^{jk} z_{s+(n-j-1)k} \qquad (|\alpha| \leqq q)$$

satisfies the inequality

$$|u_n| \leqq M\mu_1(q+\epsilon)^n, \tag{6.14}$$

where μ_1 is independent of $\{z_n\}$.

We see at once that

$$|u_r| \leqq M(q+\epsilon)^r \qquad (r \not\equiv s).$$

Also,

$$|u_{s+nk}| \leqq M\sum_{j=0}^{n-1} |\alpha|^{jk}(q+\epsilon)^{s+(n-j-1)k} \leqq \frac{M}{(q+\epsilon)^k - |\alpha|^k}(q+\epsilon)^{s+nk}.$$

We may therefore choose

$$\mu_1 = \max\left\{1, \frac{1}{(q+\epsilon)^k - |\alpha|^k}\right\}. \tag{6.15}$$

LEMMA 6.3. *Let $\mathcal{H}$ be k-periodic and admissible (in $|t| \leqq q$), and let $\epsilon > 0$ be chosen small enough so that $\mathcal{H}$ is also admissible in $|t| \leqq q+\epsilon$. There exists a constant μ_2 independent of $\{z_n\}$ such that if $|z_n| \leqq M(q+\epsilon)^n$ then*

$$|(\mathcal{H})_n[\{z_j\}]| \leqq M\mu_2(q+\epsilon)^n. \tag{6.16}$$

If we expand $H_n(t)$:

$$H_n(t) = \sum_{j=0}^{\infty} h_{nj}t^j,$$

then $\theta' < 1/(q+\epsilon)$ and N exist such that $|h_n| \leqq N\theta'^j$. Hence

$$|(\mathcal{K})_n[\{z_j\}]| \leqq NM\sum_{j=0}^{\infty}\theta'^{j}(q+\epsilon)^{n+j} = \frac{MN}{1-\theta'(q+\epsilon)}(q+\epsilon)^n,$$

so we may choose

$$\mu_2 = \frac{N}{1-\theta'(q+\epsilon)}. \tag{6.17}$$

If we apply Lemmas 6.2, 6.3 at each step in the proof of Lemma 6.1 we obtain the following lemma.

LEMMA 6.4. *There is a constant μ independent of $\{z_n\}$ such that the inequality*

$$|({}^{(l)}V_l[\{z_j\}])_n| \leqq M\mu(q+\epsilon)^n \qquad (n = 0, 1, \cdots) \tag{6.18}$$

holds whenever $|z_n| \leqq M(q+\epsilon)^n$.

REMARK. We wish to make clear the notation used on the left side of (6.18), *as applied to* ${}^{(l)}V_l$, particularly since we shall be using it freely. The subscript n indicates that we are dealing with the element of index n in a *sequence.* Let T: $\{t_n\}$ denote this sequence: $T = {}^{(l)}V_l[\{z_j\}]$. Then $\{t_n\}$ is that *unique* solution of the system

$$(\mathcal{A})_n[T] = z_n \qquad (n = 0, 1, \cdots)$$

obtained by choosing all λ's zero in the process of Lemma 6.1.

LEMMA 6.5. *Let $\mathcal{A}^*: A_n^*(t) = \sum_0^\infty a_{nj}^* t^j$ be admissible, so that θ and $\{k_n\}$ exist such that*

$$|a_{nj}^*| \leqq k_n\theta^j \qquad [\theta < 1/q;\ k_n \downarrow 0].$$

If $|z_n| \leqq M(q+\epsilon)^n$, where $\epsilon > 0$ is small enough so that $\theta < 1/(q+\epsilon)$, then

$$|(\mathcal{A}^*)_n[\{z_j\}]| \leqq \frac{Mk_n}{1-\theta(q+\epsilon)}(q+\epsilon)^n. \tag{6.19}$$

The proof is immediate, since $(\mathcal{A}^*)_n[\{z_j\}] = \sum_{j=0}^\infty a_{nj}^* z_{n+j}$.

Let

$$\beta = \frac{k_0}{1-\theta(q+\epsilon)}. \tag{6.20}$$

It is no restriction to assume as we do that the following inequalities hold:

$$\beta < 1, \qquad 0 < \frac{\beta\mu}{1-\beta\mu} < 1, \tag{6.21}$$

since they can always be achieved by a suitable truncation. Here μ is the constant of Lemma 6.4.

Write equations (6.13) in the form

$$(\mathcal{A})_n[X] \equiv (\mathcal{B}_1\mathcal{C}_2 \cdots \mathcal{B}_l\mathcal{C}_{l+1})_n[X] = c_n - (\mathcal{A}^*)_n[X]. \tag{6.22}$$

We set up a sequence of approximations $X^{(r)}$, $r=1, 2, \cdots$, defined by

$$(\mathcal{B}_1\mathcal{C}_2 \cdots \mathcal{B}_l\mathcal{C}_{l+1})_n[X^{(r)}] = c_n - (\mathcal{A}^*)_n[X^{(r-1)}], \tag{6.23}$$

with $X^{(-1)} \equiv \{0\}$. By Lemma 6.1 we can write

$$X^{(1)} = {}^{(l)}V_l[\{c_j\}] + \sum_{p=1}^{l} \lambda_p \, {}^{(l)}V_{l-p},$$

where ${}^{(l)}V_l[\{c_j\}]$ has the meaning ascribed to it in the remark following Lemma 6.4.

If we define $\{c_n^{(1)}\}$, $\{\delta_{p,n}^{(1)}\}$ by

$$c_n^{(1)} = c_n - (\mathcal{A}^*)_n[{}^{(l)}V_l[\{c_j\}]]; \qquad \delta_{p,n}^{(1)} = (\mathcal{A}^*)_n[{}^{(l)}V_{l-p}], \tag{6.24}$$

then

$$(\mathcal{A})_n[X^{(2)}] = c_n^{(1)} - \sum_{p=1}^{l} \lambda_p \delta_{p,n}^{(1)};$$

so

$$X^{(2)} = {}^{(l)}V_l[\{c_j^{(1)}\}] - \sum_{p=1}^{l} \lambda_p \, {}^{(l)}V_l[\{\delta_{p,j}^{(1)}\}] + \sum_{p=1}^{l} \lambda_p \, {}^{(l)}V_{l-p}.$$

And in general, setting

$$c_n^{(r)} \equiv c_n - (\mathcal{A}^*)_n[{}^{(l)}V_l[\{c_j^{(r-1)}\}]]; \quad \delta_{p,n}^{(r)} \equiv (\mathcal{A}^*)_n[{}^{(l)}V_l[\{\delta_{p,j}^{(r-1)}\}]], \tag{6.25}$$

then

$$\begin{aligned} X^{(r)} = {} & {}^{(l)}V_l[\{c_j^{(r-1)}\}] \\ & + \sum_{p=1}^{l} \lambda_p \, {}^{(l)}V_{l-p} - \sum_{p=1}^{l} \lambda_p \left\{ \sum_{m=1}^{r-1} (-1)^{m+1} \, {}^{(l)}V_l[\{\delta_{p,j}^{(m)}\}] \right\}, \end{aligned} \tag{6.26}$$

and

$$(\mathcal{A})_n[X^{(r+1)}] = c_n^{(r)} + \sum_{p=1}^{l} \lambda_p \left\{ \sum_{m=1}^{r} (-1)^m \delta_{p,n}^{(m)} \right\}. \tag{6.27}$$

Let $X_0^{(r)}$ be the value of $X^{(r)}$ when all λ's are chosen to be zero:

$$X_0^{(r)} = {}^{(l)}V_l[\{c_j^{(r-1)}\}] \qquad (c_n^{(0)} \equiv c_n). \tag{6.28}$$

Since $((c_n)) \leqq q$, there is an $M = M(\epsilon)$ such that $|c_n| \leqq M(q+\epsilon)^n$. Hence if $X_0^{(r)} \equiv \{x_{0,n}^{(r)}\}$, then by Lemma 6.4,

$$| x_{0,n}^{(1)} | \leqq M\mu(q+\epsilon)^n;$$

and from Lemma 6.5, and (6.25) for $r=1$,

$$| c_n^{(1)} | \leqq M(1+\mu\beta)(q+\epsilon)^n.$$

This gives us the inequalities

$$| x_{0,n}^{(2)} | \leqq M\mu(1+\mu\beta)(q+\epsilon)^n; \qquad | c_n^{(2)} | \leqq M(q+\epsilon)^n\{1+\mu\beta+(\mu\beta)^2\};$$

and a simple induction leads to

$$(6.29) \quad \begin{aligned} | c_n^{(r)} | &\leqq M(q+\epsilon)^n\{1+(\mu\beta)+\cdots+(\mu\beta)^r\} \leqq \frac{M}{1-\mu\beta}(q+\epsilon)^n; \\ | x_{0,n}^{(r)} | &\leqq M\mu(q+\epsilon)^n\{1+(\mu\beta)+\cdots+(\mu\beta)^{r-1}\} \leqq \frac{M\mu}{1-\mu\beta}(q+\epsilon)^n. \end{aligned}$$

Since $X_0^{(r)}$ and $\{c_n^{(r)}\}$ are independent of ϵ we see that these sequences are all of type not exceeding q. Now

$$(6.30) \qquad X_0^{(r)} - X_0^{(r-1)} = {}^{(l)}V_l[\{c_j^{(r-1)} - c_j^{(r-2)}\}],$$

and

$$(6.31) \quad \begin{aligned} c_n^{(r-1)} - c_n^{(r-2)} &= -(\mathcal{A}^*)_n[{}^{(l)}V_l[\{c_j^{(r-2)} - c_j^{(r-3)}\}]] \\ &= -(\mathcal{A}^*)_n[X_0^{(r-1)} - X_0^{(r-2)}]. \end{aligned}$$

From (6.31)

$$| c_n^{(1)} - c_n | = | -(\mathcal{A}^*)_n[{}^{(l)}V_l[\{c_j\}]] | \leqq M\mu\beta(q+\epsilon)^n,$$

so

$$| x_{0,n}^{(2)} - x_{0,n}^{(1)} | \leqq M\mu^2\beta(q+\epsilon)^n.$$

By induction we obtain

$$(6.32) \qquad | x_{0,n}^{(r)} - x_{0,n}^{(r-1)} | \leqq M\mu^r\beta^{r-1}(q+\epsilon)^n.$$

Since $\mu\beta < 1$ by (6.21), it follows that

$$(6.33) \qquad x_{0,n} \equiv \lim_{r\to\infty} x_{0,n}^{(r)} \qquad (\{x_{0,n}\} \equiv X_0)$$

exists, $n = 0, 1, \cdots$; and (6.29) shows that X_0 is admissible, with

$$| x_{0,n} | \leqq \frac{M\mu}{1-\mu\beta}(q+\epsilon)^n.$$

We wish to show that X_0 satisfies (6.22). For $p>r$,

$$\left| x_{0,n}^{(p)} - x_{0,n}^{(r)} \right| \leqq \frac{M\mu}{1-\mu\beta} (\mu\beta)^r (q+\epsilon)^n;$$

and letting $p \to \infty$:

$$\left| x_{0,n} - x_{0,n}^{(r)} \right| \leqq \frac{M\mu}{1-\mu\beta} (\mu\beta)^r (q+\epsilon)^n. \tag{6.34}$$

We have

$$(\mathcal{A}+\mathcal{A}^*)_n[X_0] = (\mathcal{A}+\mathcal{A}^*)_n[X_0 - X_0^{(r)}] + (\mathcal{A}+\mathcal{A}^*)_n[X_0^{(r)}].$$

Also, from (6.23),

$$(\mathcal{A})_n[X_0^{(r)}] = c_n - (\mathcal{A}^*)_n[X_0^{(r-1)}],$$

as we see by taking all λ's zero. Hence

$$(\mathcal{A}+\mathcal{A}^*)_n[X_0] - c_n = (\mathcal{A}+\mathcal{A}^*)_n[X_0 - X_0^{(r)}] + (\mathcal{A}^*)_n[X_0^{(r)} - X_0^{(r-1)}].$$

From (6.34) and Lemmas 6.3 and 6.5, there is a constant μ_3 independent of M, μ, β such that

$$\left| (\mathcal{A}+\mathcal{A}^*)_n[X_0 - X_0^{(r)}] \right| \leqq \frac{M\mu\mu_3}{1-\beta\mu} (\beta\mu)^r (q+\epsilon)^n;$$

and from (6.32) and Lemma 6.5,

$$\left| (\mathcal{A}^*)_n[X_0^{(r)} - X_0^{(r-1)}] \right| \leqq M(\mu\beta)^r (q+\epsilon)^n.$$

Hence $C=C(\epsilon)$ exists, independent of r and n, such that

$$\left| (\mathcal{A}+\mathcal{A}^*)_n[X_0] - c_n \right| \leqq C(\beta\mu)^r (q+\epsilon)^n;$$

so on letting $r \to \infty$ we see that

$$(\mathcal{A}+\mathcal{A}^*)_n[X_0] = c_n.$$

That is, X_0 *is* a solution of (6.22).

Let us return to (6.26). Denote the coefficience of λ_p by $X_p^{(r)}$, so that

$$X^{(r)} = X_0^{(r)} + \sum_{p=1}^{l} \lambda_p X_p^{(r)}, \tag{6.35}$$

with

$$X_p^{(r)} = {}^{(l)}V_{l-p} + \sum_{m=1}^{r-1} (-1)^m \cdot {}^{(l)}V_l[\{\delta_{p,i}^{(m)}\}]. \tag{6.36}$$

We wish to show that as $r \to \infty$, $X_p^{(r)}$ has a limiting value X_p, that this sequence is admissible, and that it satisfies the homogeneous equation.

Now ${}^{(l)}V_{l-p}$ $(p=1, \cdots, l)$ are particular solutions of the homogeneous system $(\mathcal{A})_n[X]=0$, so there is a number N such that $|({}^{(l)}V_{l-p})_n| \leqq N(q+\epsilon)^n$. From (6.24),

$$|\delta_{p,n}^{(1)}| \leqq N\beta(q+\epsilon)^n;$$

and if we use (6.25), an induction gives the inequality

$$|\delta_{p,n}^{(r)}| \leqq N\beta(\beta\mu)^{r-1}(q+\epsilon)^n. \tag{6.37}$$

It follows from (6.36) that

$$X_p = \lim_{r\to\infty} X_p^{(r)} \qquad (p = 1, \cdots, l) \tag{6.38}$$

exists and has the value

$$X_p = {}^{(l)}V_{l-p} + \sum_{m=1}^{\infty} (-1)^m \, {}^{(l)}V_l[\{\delta_{p,j}^{(m)}\}] \qquad (p = 1, \cdots, l). \tag{6.39}$$

Moreover, if $X_p \equiv \{x_{p,n}\}$, then

$$|x_{p,n}| \leqq N(q+\epsilon)^n \sum_{m=1}^{\infty} (\beta\mu)^m = \frac{N}{1-\beta\mu}(q+\epsilon)^n. \tag{6.40}$$

$X_p(p=1, \cdots, l)$ is thus admissible.

We now have need of the following lemma.

LEMMA 6.6. *The following interchanges of order of operations are permissible*:

$$\begin{aligned}(\mathcal{A}^*)_n\left[\sum_{m=1}^{\infty}(-1)^m \, {}^{(l)}V_l[\{\delta_{p,j}^{(m)}\}]\right] &= \sum_{m=1}^{\infty}(-1)^m(\mathcal{A}^*)_n[{}^{(l)}V_l[\{\delta_{p,j}^{(m)}\}]],\\ (\mathcal{A})_n\left[\sum_{m=1}^{\infty}(-1)^m \, {}^{(l)}V_l[\{\delta_{p,j}^{(m)}\}]\right] &= \sum_{m=1}^{\infty}(-1)^m(\mathcal{A})_n[{}^{(l)}V_l[\{\delta_{p,j}^{(m)}\}]].\end{aligned} \tag{6.41}$$

As for the first of these, this amounts to a change of order of summation of the double series

$$\sum_{p=0}^{\infty}\sum_{m=1}^{\infty} a_{np}^*(-1)^m({}^{(l)}V_l[\{\delta_{p,j}^{(m)}\}])_{n+p},$$

which is valid because of the absolute convergence of this series (as seen from (6.37)). The second case requires more consideration.

Let it be recalled that ${}^{(l)}V_l[\{\gamma_j\}]$ is a definite solution of the system

$$(\mathcal{A})_n[X] = \gamma_n \qquad (n = 0, 1, \cdots),$$

namely that solution obtained by taking all λ's as zero in the process of

Lemma 6.1. Turning to that process, denote by ${}^{(1)}U_1^{(m)}$, ${}^{(1)}U_1$ those solutions of (6.5) corresponding respectively to

$$\{\gamma_n\} = \{\delta_{p,n}^{(m)}\}, \qquad \{\gamma_n\} = \left\{\sum_1^\infty (-1)^m \delta_{p,n}^{(m)}\right\},$$

for which $\lambda_1=0$. These solutions are given by

$$^{(1)}u_{1,r} = \gamma_r \quad (r \neq s); \qquad {}^{(1)}u_{1,s+nk} = \sum_{j=0}^{n-1} \alpha_1^{jk}\gamma_{s+(n-j-1)k}.$$

It is clear that we have

$$^{(1)}U_1 = \sum_{m=1}^\infty (-1)^m \cdot {}^{(1)}U_1^{(m)}. \tag{6.42}$$

Now define ${}^{(1)}V_1^{(m)}$, ${}^{(1)}V_1$ as the unique solutions of

$$(\mathcal{C}_2)_n[{}^{(1)}V_1^{(m)}] = {}^{(1)}U_1^{(m)}; \qquad (\mathcal{C}_2)_n[{}^{(1)}V_1] = {}^{(1)}U_1.$$

Then

$$({}^{(1)}V_1^{(m)})_n = (\mathcal{C}_2^{-1})_n[{}^{(1)}U_1^{(m)}]; \qquad ({}^{(1)}V_1)_n = (\mathcal{C}_2^{-1})_n[{}^{(1)}U_1];$$

and the relation

$$^{(1)}V_1 = \sum_1^\infty (-1)^m \cdot {}^{(1)}V_1^{(m)} \tag{6.43}$$

now follows from the absolute convergence of the double series that $(\mathcal{C}_2^{-1})_n[\sum_{m=1}^\infty (-1)^m \cdot {}^{(1)}U_1^{(m)}]$ represents for each n.

Proceeding in this manner, we find that (6.42) and (6.43) continue to hold when the superscript and subscript (1) is replaced by (2), (3), $\cdots$, (l). The case (l) reads as follows:

$$^{(l)}V_l\left[\left\{\sum_{m=1}^\infty (-1)^m \delta_{p,j}^{(m)}\right\}\right] = \sum_{m=1}^\infty (-1)^m \cdot {}^{(l)}V_l[\{\delta_{p,j}^{(m)}\}]. \tag{6.44}$$

Applying $\mathcal{A}$ to both sides (as can be done), and recalling the meaning of ${}^{(l)}V_l[\{z_j\}]$, we get

$$
\begin{aligned}
(\mathcal{A})_n\left[\sum_1^\infty (-1)^m\, {}^{(l)}V_l[\{\delta_{p,j}^{(m)}\}]\right]
&= (\mathcal{A})_n\left[{}^{(l)}V_l\left[\left\{\sum_{m=1}^\infty (-1)^m \delta_{p,j}^{(m)}\right\}\right]\right] \\
&= \sum_{m=1}^\infty (-1)^m \delta_{p,n}^{(m)} = \sum_1^\infty (-1)^m (\mathcal{A})_n[{}^{(l)}V_l[\{\delta_{p,j}^{(m)}\}]];
\end{aligned}
$$

and this is the second relation of (6.41).

Now we can show that X_p ($p=1, \cdots, l$) is a solution of the homogeneous system

$$(6.45) \qquad (\mathcal{A}+\mathcal{A}^*)_n[X_p] = 0 \qquad (n = 0, 1, \cdots).$$

From (6.39),

$$(\mathcal{A}+\mathcal{A}^*)_n[X_p] = (\mathcal{A})_n[{}^{(l)}V_{l-p}] + (\mathcal{A}^*)_n[{}^{(l)}V_{l-p}]$$
$$+ (\mathcal{A}+\mathcal{A}^*)_n\left[\sum_{m=1}^{\infty}(-1)^m\, {}^{(l)}V_l[\{\delta_{p,j}^{(m)}\}]\right].$$

Now

$$(\mathcal{A})_n[{}^{(l)}V_{l-p}] = 0,$$

as we know from Lemma 6.1; and from (6.24),

$$(\mathcal{A}^*)_n[{}^{(l)}V_{l-p}] = \delta_{p,n}^{(1)}.$$

Hence on applying Lemma 6.6, and (6.25),

$$(\mathcal{A}+\mathcal{A}^*)_n[X_p]$$
$$= \delta_{p,n}^{(1)} + \sum_{m=1}^{\infty}(-1)^m(\mathcal{A})_n[{}^{(l)}V_l[\{\delta_{p,j}^{(m)}\}] + (\mathcal{A}^*)_n]\left[\sum_{m=1}^{\infty}(-1)^m\, {}^{(l)}V_l[\{\delta_{p,j}^{(m)}\}]\right]$$
$$= \delta_{p,n}^{(1)} + \sum_{m=1}^{\infty}(-1)^m\delta_{p,n}^{(m)} + \sum_{m=1}^{\infty}(-1)^m\delta_{p,n}^{(m+1)} = 0.$$

That is, (6.45) holds.

So far we have established the existence of a particular admissible solution X_0 of the nonhomogeneous system of equations (6.13), and l admissible solutions X_p ($p=1, \cdots, l$) of the homogeneous system. There remain two tasks: (i) To show that $X_1, \cdots, X_l$ are linearly independent; and (ii) To prove that l is the maximum number of linearly independent solutions, so that if $Y=\{y_n\}$ is a solution of (6.13), then Y is the sum of X_0 and a linear combination of $X_1, \cdots, X_l$. These we take up in turn.

Let $\sigma_1, \cdots, \sigma_l$ be arbitrary constants, once chosen fixed, and not all zero; and set

$$(6.46) \qquad V \equiv \sum_{p=1}^{l}\sigma_p\, {}^{(l)}V_{l-p}.$$

Then $V\equiv\{v_n\}$ is a *nonzero* solution of

$$(\mathcal{A})_n[V] = 0 \qquad (n = 0, 1, \cdots),$$

since ${}^{(l)}V_{l-p}$ ($p=1, \cdots, l$) are linearly independent (as seen in Lemma 6.1).

There therefore exists a *positive* constant $H=H(\epsilon)$ such that

(6.47) $$|v_n| \leqq H(q+\epsilon)^n$$

and such that for at least one value $n=n'$,

(6.48) $$|v_{n'}| = H(q+\epsilon)^{n'}.$$

From (6.39),

(6.49) $$\sum_{p=1}^{l} \sigma_p X_p = V + \sum_{r=1}^{\infty} (-1)^r \, {}^{(l)}V_l\left[\left\{\sum_{p=1}^{l} \sigma_p \delta_{p,j}^{(r)}\right\}\right].$$

Now from (6.24), $\delta_{p,n}^{(1)} = (\mathcal{A}^*)_n[{}^{(l)}V_{l-p}]$, so

$$\sum_{p=1}^{l} \sigma_p \delta_{p,n}^{(1)} = (\mathcal{A}^*)_n[V];$$

hence

$$\left|\sum_{p=1}^{l} \sigma_p \delta_{p,n}^{(1)}\right| \leqq H\beta(q+\epsilon)^n.$$

Consequently

$$\left|\left({}^{(l)}V_l\left[\left\{\sum_{p=1}^{l} \sigma_p \delta_{p,j}^{(1)}\right\}\right]\right)_n\right| \leqq H\beta\mu(q+\epsilon)^n;$$

and from (6.25),

$$\left|\sum_{p=1}^{l} \sigma_p \delta_{p,n}^{(2)}\right| \leqq H\beta^2\mu(q+\epsilon)^n,$$

and so on. A simple induction yields the inequality

(6.50) $$\left|\left({}^{(l)}V_l\left[\left\{\sum_{p=1}^{l} \sigma_p \delta_{p,j}^{(r)}\right\}\right]\right)_n\right| \leqq H(\beta\mu)^r(q+\epsilon)^n.$$

Thus, since $\beta\mu<1$,

(6.51) $$\sum_{r=1}^{\infty}\left|(-1)^r\left({}^{(l)}V_l\left[\left\{\sum_{p=1}^{l} \sigma_p \delta_{p,j}^{(r)}\right\}\right]\right)_n\right| \leqq \frac{H\beta\mu}{1-\beta\mu}(q+\epsilon)^n.$$

But we also have $\beta\mu/(1-\beta\mu)<1$ by (6.21), so examination of (6.49), in view of (6.48) and (6.51), shows that

$$\left(\sum_{p=1}^{l} \sigma_p X_p\right)_{n'} \neq 0.$$

The solution $\sum_{p=1}^{l} \sigma_p X_p$ is thus *not* the zero sequence, and we conclude from the fact that the σ's may assume any values (not all zero) that $X_1, \cdots, X_l$ are *linearly independent* solutions of the homogeneous system

$$(\mathcal{A}+\mathcal{A}^*)_n[X] = 0.$$

There now remains, as stated before, to show that every admissible solution Y of (6.13) has the expression

$$Y = X_0 + \sum_{p=1}^{l} \lambda_p X_p \tag{6.52}$$

for proper choice of the λ's. To this effect, let X be defined by

$$X = X_0 + \sum_{p=1}^{l} \lambda_p X_p,$$

with as yet undetermined coefficients $\{\lambda_p\}$. Then

$$(\mathcal{A}+\mathcal{A}^*)_n[Y] = (\mathcal{A}+\mathcal{A}^*)_n[X] =. c_n \qquad (n = 0, 1, \cdots);$$

and on setting

$$Y' = Y - X_0 \tag{6.53}$$

we have $(\mathcal{A}+\mathcal{A}^*)_n[Y']=0$.

Write this equation as $(\mathcal{A})_n[Y'] = -(\mathcal{A}^*)_n[Y']$. From Lemma 6.1 there exist constants $\lambda_p = \lambda_p'$ $(p=1, \cdots, l)$ such that

$$Y' = {}^{(l)}V_{l;Y'} + \sum_{p=1}^{l} \lambda_p' \; {}^{(l)}V_{l-p}, \tag{6.54}$$

where ${}^{(l)}V_{l;Y'}$ is the solution ${}^{(l)}V_l$ of Lemma 6.1 corresponding to

$$\{\gamma_n\} = \{-(\mathcal{A}^*)_n[Y']\}:$$

$$(\mathcal{A})_n[{}^{(l)}V_{l;Y'}] = -(\mathcal{A}^*)_n[Y'],$$

$${}^{(l)}V_{l;Y'} = {}^{(l)}V_l[\{-(\mathcal{A}^*)_n[Y']\}].$$

Define X', Z by

$$X' = \sum_{p=1}^{l} \lambda_p' X_p = \sum_{p=1}^{l} \lambda_p' \; {}^{(l)}V_{l-p} + \sum_{r=1}^{\infty} (-1)^r \, {}^{(l)}V_l\left[\left\{\sum_{p=1}^{l} \lambda_p' \delta_{p,j}^{(r)}\right\}\right], \tag{6.55}$$

$$\begin{aligned} Z = Y' - X' &= {}^{(l)}V_{l;Y'} - \sum_{r=1}^{\infty} (-1)^r \, {}^{(l)}V_l\left[\left\{\sum_{p=1}^{l} \lambda_p' \delta_{p,j}^{(r)}\right\}\right] \\ &= {}^{(l)}V_l[\{-(\mathcal{A}^*)_n[Y']\}] \\ &\quad - {}^{(l)}V_l\left[\left\{\sum_{p=1}^{l} \lambda_p' \left(\sum_{r=1}^{\infty} (-1)^r \delta_{p,j}^{(r)}\right)\right\}\right], \end{aligned} \tag{6.56}$$

the last expression following from the proof of Lemma 6.6.

Again by Lemma 6.6,

$$\begin{aligned}(\mathcal{A}^*)_n[X'] &= \sum_{p=1}^{l} \lambda_p' (\mathcal{A}^*)_n[{}^{(l)}V_{l-p}] \\ &\quad + \sum_{r=1}^{\infty} (-1)^r (\mathcal{A}^*)_n\left[{}^{(l)}V_l\left[\left\{\sum_{p=1}^{l} \lambda_p' \delta_{p,j}^{(r)}\right\}\right]\right] \\ &= \sum_{p=1}^{l} \lambda_p' \delta_{p,n}^{(1)} + \sum_{r=1}^{\infty} (-1)^r \left(\sum_{p=1}^{l} \delta_{p,n}^{(r+1)}\right) \\ &= \sum_{p=1}^{l} \lambda_p' \left(\sum_{r=1}^{\infty} (-1)^{r-1} \delta_{p,n}^{(r)}\right).\end{aligned}$$

Hence

$$Z = {}^{(l)}V_l[\{-(\mathcal{A}^*)_j[Y']\}] + {}^{(l)}V_l[\{(\mathcal{A}^*)_j[X']\}] = {}^{(l)}V_l[\{(\mathcal{A}^*)_j[X'-Y']\}]$$

or

$$Z = -{}^{(l)}V_l[\{(\mathcal{A}^*)_j[Z]\}]. \tag{6.57}$$

Since Z is admissible, there is a constant $Q=Q(\epsilon)$ such that

$$|z_n| \leqq Q(q+\epsilon)^n,$$

and such that Q is as small as possible, so that an index $n=n'$ exists for which

$$|z_{n'}| = Q(q+\epsilon)^{n'}. \tag{6.58}$$

From Lemma 6.5,

$$|(\mathcal{A}^*)_n[Z]| \leqq Q\beta(q+\epsilon)^n,$$

so

$$|({}^{(l)}V_l[\{(\mathcal{A}^*)_j[Z]\}])_n| \leqq Q\beta\mu(q+\epsilon)^n.$$

Since $\beta\mu<1$, we see from (6.57) that

$$|z_{n'}| \leqq Q\beta\mu(q+\epsilon)^{n'}.$$

This is in contradiction to (6.58) unless $Q=0$; which means that $z_n\equiv 0$; that is, that $Z'=Y'-X'\equiv\{0\}$. Thus, $Y'=X'$, and

$$Y = X_0 + \sum_{p=1}^{l} \lambda_p' X_p,$$

as was to be shown.

We may now sum up:

THEOREM 6.1. *Let $\mathcal{A}+\mathcal{A}^*$ be admissible (in $|t|\leqq q$) with $\mathcal{A}$ of order l, and* *consider the system of equations* (1.6). *This system has an admissible particular solution* X_0, *and the homogeneous system* ($c_n \equiv 0$) *has precisely* l *linearly independent admissible solutions* $(X_1, \cdots, X_l)$. *The most general admissible solution of* (1.6) *is*

$$X = X_0 + \sum_{p=1}^{l} \lambda_p X_p, \tag{6.59}$$

for arbitrary choice of the λ's.

To this we can add the following theorem.

THEOREM 6.2. *Let the hypotheses of Theorem* 6.1 *hold. If* $((c_n)) = d \leqq q$, *there is a solution* X_0' *of* (1.6) *of type not exceeding* d, *and the most general solution whose type is not greater than* d *has the form*

$$X = X_0' + \sum_{p=1}^{h} \lambda_p' X_p'$$

where $\mathcal{A}$ *is of order* h *in* $|t| \leqq d$ *and* X_p' $(p = 1, \cdots, h)$ *is a set of linearly independent solutions of the homogeneous equation, of type not exceeding* d. *Let the* l *nests of zeros* $\{\omega^f \alpha_j\}$, $j = 1, \cdots, l$, *be grouped into classes, all those nests having a common absolute value being put into the same class. Let there be* s *classes, of absolute value*

$$0 < \mu_1 < \cdots < \mu_s,$$

and let μ_j *correspond to* g_j *nests, so that* $\sum_{j=1}^{s} g_j = l$. *Then a fundamental set of solutions of the homogeneous system* (1.6) ($c_n \equiv 0$) *can be found such that exactly* g_j *of these solutions are of type* μ_j $(j = 1, \cdots, s)$.

The first part of the theorem is obvious on considering $\mathcal{A}$ as admissible and of order h in $|t| \leqq d$. Consider therefore the homogeneous case. If $q_1 < \mu_1$, then $\mathcal{A}$ is of zero order in $|t| \leqq q_1$, so there is no solution X of the homogeneous system, other than $X \equiv \{0\}$, that is of type less than μ_1. In $|t| \leqq \mu_1$, $\mathcal{A}$ is of order g_1, so by Theorem 6.1 there are exactly g_1 linearly independent solutions of type not exceeding μ_1 and therefore of type equal to μ_1. If q_2 is chosen in $\mu_1 < q_2 < \mu_2$, $\mathcal{A}$ is still of order g_1 in $|t| \leqq q_2$, so no new solutions are brought in until we consider $\mathcal{A}$ in $|t| \leqq \mu_2$. For this case $\mathcal{A}$ is of order $g_1 + g_2$. Having already accounted for g_1 solutions, we now have g_2 new solutions (all $g_1 + g_2$ solutions being linearly independent), and these are of necessity of type equal to μ_2; and so on, until the complete statement of the theorem is established.

THE PENNSYLVANIA STATE COLLEGE
STATE COLLEGE, PA.